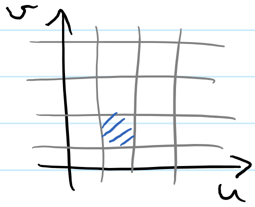
L18: More on Change of Variables in Double Integrals

October 24, 2016 12:31 PM

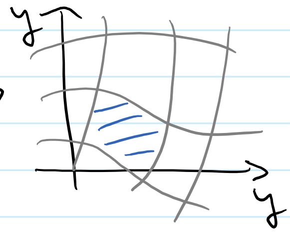
Midterm tomorrow → Stirling A 6pm–8pm
↳ do practice midterm!

More on Change of Variables in Double Integrals

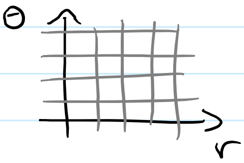


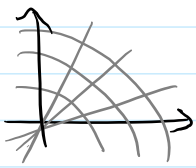
$\xrightarrow{T}$

$(u,v) \mapsto (x(u,v), y(u,v))$



Example (Polar Coordinates):



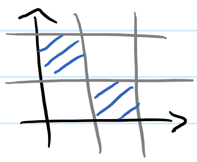


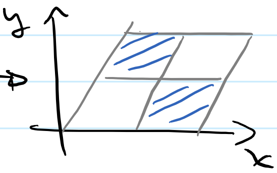
Local Picture:

Examples of linear transformations:

$\mathbb{R}^2_{(u,v)} \to \mathbb{R}^2_{(x,y)}$

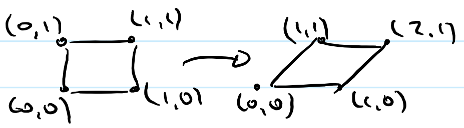
- Shear $(u,v) \mapsto (u+v, v)$





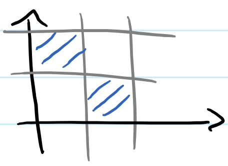
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

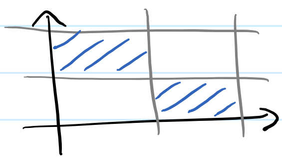
$\det = 1 - 0 = 1$



Area = 1

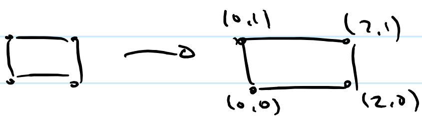
- Scaling $(u,v) \mapsto (2u, v)$





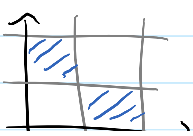
$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

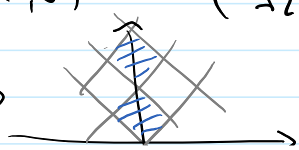
$\det = 2 - 0 = 2$



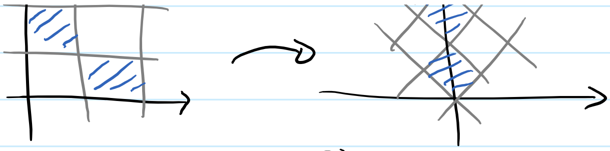
Area = 2

- Rotation $(u,v) \mapsto \left(\frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}}\right)$



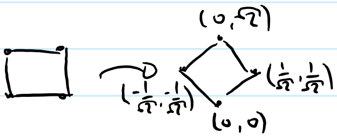


$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

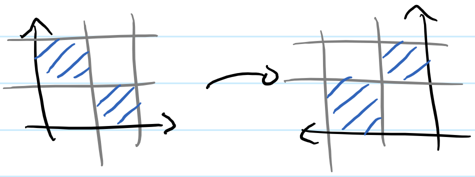


$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\det = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1$$

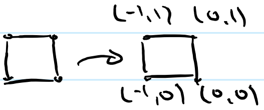


- Reflection $(u,v) \mapsto (-u,v)$



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det = -1$



Area = 1

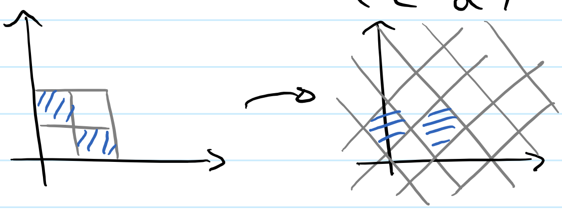
A general linear transformation

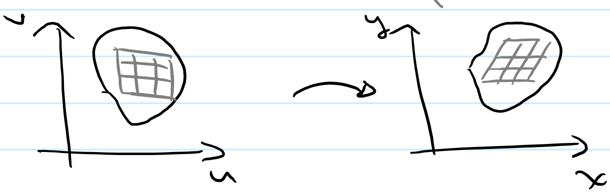
$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$

$$T(c\vec{u}) = cT(\vec{u})$$

In linear algebra, show that every linear transformation can be represented by a matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (u,v) \mapsto (au+bv,\ cu+dv)$$





Theorem:

Let $T: \mathbb{R}^2_{(u,v)} \mapsto \mathbb{R}^2_{(x,y)}$ be a linear transformation, $\det T \neq 0$. Let R^* be a region in $\mathbb{R}^2_{(u,v)}$ and $R = T(R^*)$

$$\iint_R f(x,y)\,dA = \iint_{R^*} f(x(u,v), y(u,v))\,|\det T|\,dA$$

Example:

Area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

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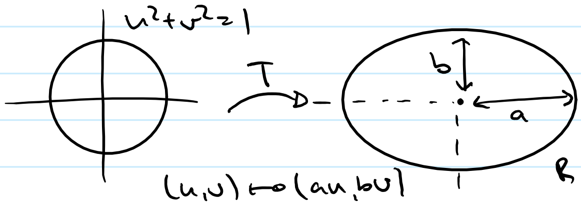
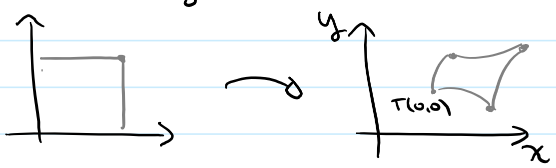


By change of variables theorem:

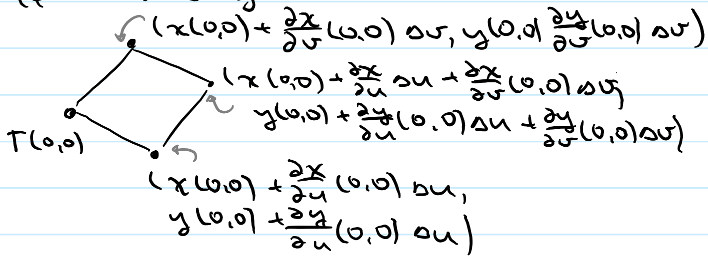
$$\iint_R 1\, dA = \iint_{R^*} 1 \left|\det\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\right| dA$$
$$= ab \iint_{R^*} 1\, dA$$
$$= ab\pi$$

Global Picture

T no longer linear.



Approximate by:



Conclusion:

T is well-approximated near (0,0) by

$$\begin{pmatrix} \frac{\partial x}{\partial u}(0,0) & \frac{\partial x}{\partial v}(0,0) \\ \frac{\partial y}{\partial u}(0,0) & \frac{\partial y}{\partial v}(0,0)\end{pmatrix} =: \frac{\partial(x,y)}{\partial(u,v)}$$

This linear map is called the Jacobian of T.

Theorem:

Let T be a transformation $\mathbb{R}^2_{(u,v)} \to \mathbb{R}^2_{(x,y)}$, 1-1 in the interior of R^*

$$R = T(R^*)$$

$$\iint_R f(x,y)\, dA = \iint_{R^*} f(x(u,v), y(u,v)) \left|\det \frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

Example:

Polar coordinates:

$(u,v) \mapsto (u\cos v,\ u\sin v)$

[Other notation: $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$]

$\frac{\partial x}{\partial u} = \cos v \qquad \frac{\partial x}{\partial v} = -u\sin v$

$\frac{\partial y}{\partial u} = \sin v \qquad \frac{\partial y}{\partial v} = u\cos v$

$\frac{\partial y}{\partial u} = \sin v \qquad \frac{\partial y}{\partial v} = u\cos v$

$$\det \frac{\partial(u,v)}{\partial(x,y)} = \det \begin{pmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{pmatrix}$$

$$= u\cos^2 v - (-u\sin^2 v)$$

$$= u(\cos^2 v + \sin^2 v)$$

$$= u$$