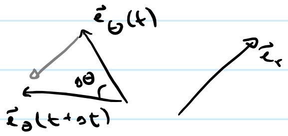
L17: Change in Variables for Double Integrals, Double Integrals in Curvilinear Coordinates

October 20, 2016 1:32 PM

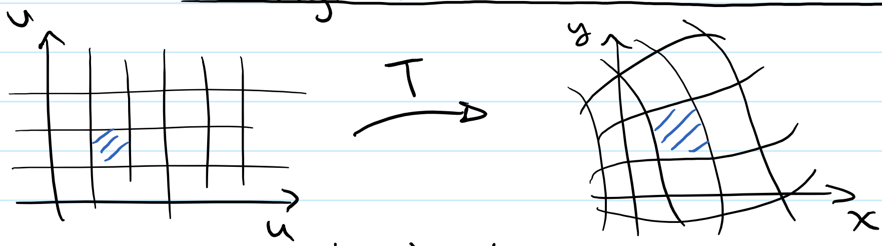
Correction: $\frac{d\vec{e}_\theta}{dt} = -\frac{d\theta}{dt}\,\vec{e}_r$, not $-\frac{dr}{dt}\,\vec{e}_r$

oops



Today: "slightly more general things" - Smirnov

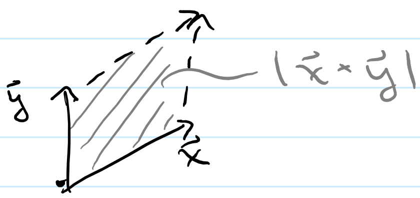
Change of Variables for Double Integrals



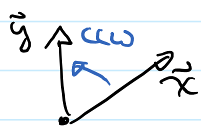
$(u,v) \mapsto (x(u,v), y(u,v))$

Cross Product in $\mathbb{R}^2$:

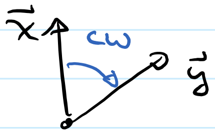
Defn: For vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^2$, their $\vec{x}\times\vec{y}$ is a real number (unlike $\mathbb{R}^3$ case!) whose magnitude is equal to the area of the parallelogram subtented by $\vec{x}$ and $\vec{y}$.



The sign of $\vec{x}\times\vec{y}$ keeps track of the orientation of $\vec{x}$ and $\vec{y}$.

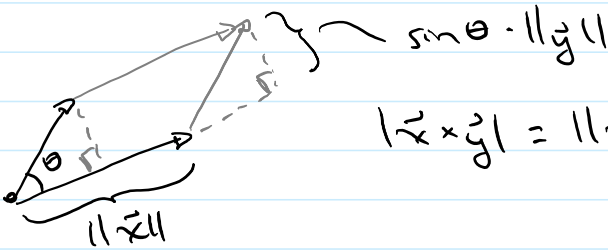


$\vec{x}\times\vec{y} > 0$



$\vec{x}\times\vec{y} < 0$

$\vec{x}\times\vec{y} = -\vec{y}\times\vec{x}$



$|\vec{x}\times\vec{y}| = \|\vec{x}\|\,\|\vec{y}\|\sin\theta$

If $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$, what is $\vec{x}\times\vec{y}$ in terms of the coordinates?

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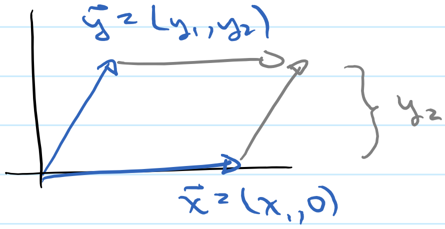


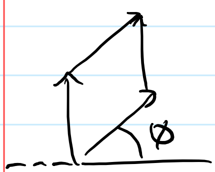
$$\vec{x} \times \vec{y} = x_1 \cdot y_2$$

The general case can be reduced to this one by the following:

<u>Geometric Principle</u>:

If T_θ is a clockwise rotation by θ radians in $\mathbb{R}^2$,
$$(T_\theta \vec{x}) \times (T_\theta \vec{y}) = \vec{x} \times \vec{y}$$

Denote the polar angle of $\vec{x}$ by ϕ.



$$\vec{x} \times \vec{y} = (T_{-\phi}\vec{x}) \times (T_{-\phi}\vec{y})$$

<u>Recall</u>: $T_{-\phi}(z_1, z_2) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$\cos\phi = \frac{x_1}{\|\vec{x}\|} \qquad \sin\phi = \frac{x_2}{\|\vec{x}\|}$$

$$= (\cos\phi\, z_1 + \sin\phi\, z_2,\ -\sin\phi\, z_1 + \cos\phi\, z_2)$$

Write: $T_{-\phi}\vec{x} = (x_1', x_2')$
$T_{-\phi}\vec{y} = (y_1', y_2')$

$$\vec{x} \times \vec{y} = (T_{-\phi}\vec{x}) \times (T_{-\phi}\vec{y}) = (x_1') \cdot (y_2')$$

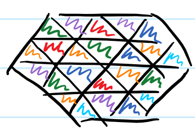
$$x_1' = \left(\frac{x_1}{\|\vec{x}\|}\right)x_1 + \left(\frac{x_2}{\|\vec{x}\|}\right)x_2 = \frac{\|\vec{x}\|^2}{\|\vec{x}\|} = \|\vec{x}\|$$

$$y_2' = \left(\frac{-x_2}{\|\vec{x}\|}\right)y_1 + \left(\frac{x_1}{\|\vec{x}\|}\right)y_2 = \frac{x_1 y_2 - x_2 y_1}{\|\vec{x}\|}$$

<u>Conclusion</u>:
$$\vec{x} \times \vec{y} = \left(\frac{x_1 y_2 - x_2 y_1}{\|\vec{x}\|}\right)\|\vec{x}\|$$

$$= x_1 y_2 - x_2 y_1$$

$$= \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$$



<u>Another derivation</u>:

$$\vec{e}_x \times \vec{e}_y = 0$$

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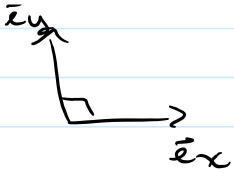


$$\vec{e}_x \times \vec{e}_y = 1 = -\vec{e}_y \times \vec{e}_x$$

$$\vec{x} \times \vec{y} = (x_1\vec{e}_x + x_2\vec{e}_y) \times (y_1\vec{e}_x + y_2\vec{e}_y)$$
$$= 0 + x_1y_2\,\vec{e}_x \times \vec{e}_y + x_2y_1\,\vec{e}_y \times \vec{e}_x + 0$$
$$= x_1y_2 - x_2y_1$$

"Assuming bilinearity"

Double Integrals in Curvilinear Coordinates.

Local Picture:

T is a linear map $\mathbb{R}^2_{(u,v)} \to \mathbb{R}^2_{(x,y)}$

$$T(u,v) = (au+bv,\ cu+dv)$$

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right] \qquad \det T \neq 0$$
$$\det T = ad - bc$$

One can check that

$$\left|\begin{array}{l} T(u+u', v+v') = T(u,v) + T(u',v') \\ T(cu, cv) = cT(u,v), \quad c \in \mathbb{R} \end{array}\right.$$

Proposition:

A linear map sends lines to lines.

Proof:

A line in $\mathbb{R}^2_{(u,v)}$ is described by the trace of
$t \mapsto \vec{r}_0 + t\vec{v}_0$, $t \in \mathbb{R}$

$$T(\vec{r}_0 + t\vec{v}_0) = T(\vec{r}_0) + tT(\vec{v}_0)$$

As t varies in $\mathbb{R}$, this again describes a line in $\mathbb{R}^2_{(x,y)}$

[$T(\vec{v}_0) \neq \vec{0}$, because $\vec{v}_0 \neq 0$ and T is invertible ($\det T \neq 0$)]

Prop^n:

a) T sends intersecting lines to intersecting lines

b) ——— parallel ———

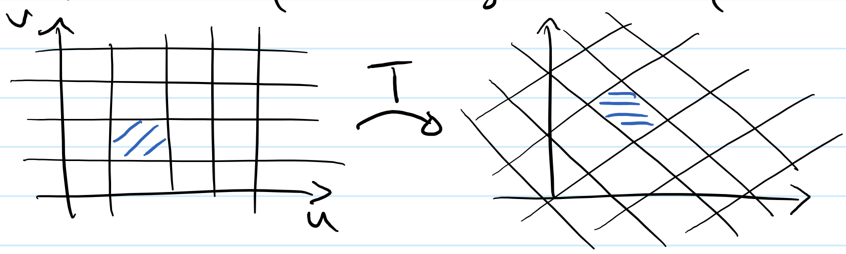
Proof:

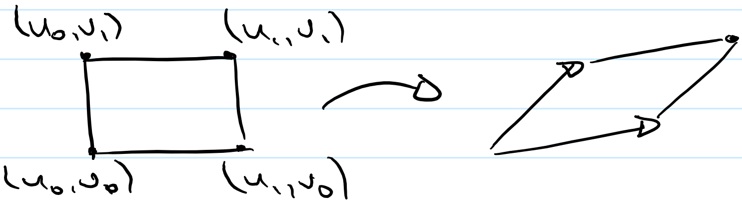
- If $\vec{x}_0$ is a point of intersection of L_1 and L_2 in $\mathbb{R}^2_{(u,v)}$, then $T(\vec{x}_0)$ is a point of intersection of $T(L_1)$ and $T(L_2)$.

• If $\vec{y}_0$ is a point of intersection of $T(L_1)$ and $T(L_2)$ then $T^{-1}(\vec{y}_0)$ is a point of intersection of L_1 and L_2.

Consequence:

T sends parallelograms to parallelograms.





The area of the parallelogram is the cross product of its subtending vectors.

$\longrightarrow = (au_1 + bv_0, cu_1 + dv_0) - (au_0 + bv_0, cu_0 + dv_0)$
$= (a(u_1 - u_0), c(u_1 - u_0))$

$\nearrow = (b(v_1 - v_0), d(v_1 - v_0))$

Their cross product is

$(a(u_1 - u_0))(d(v_1 - v_0)) - (b(v_1 - v_0))(c(u_1 - u_0))$
$= (ad - bc)(u_1 - u_0)(v_1 - v_0)$
$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \text{Area}$

The determinant of the matrix abcd measures how much T distorts area.