L16: More on Polar Coordinates

October 19, 2016 11:26 AM

Announcement: Practice Midterm is up on course page.

Midterm: Oct. 25^{th} 6-8 (next tues).

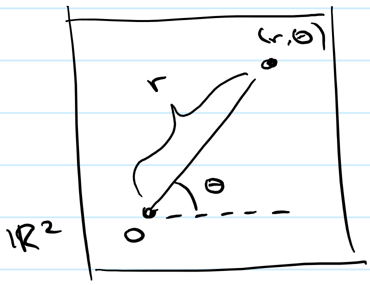
→ 6 questions

→ computation focused.

BRRR...

Today: <u>More on Polar Coordinates:</u>

Last time:



If ref. axis is the x-axis

↳ ref point is $(0,0)$

then (r,θ) is related to cartesian coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

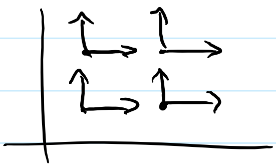
We saw how to compute:

$\int_C f\,ds$

$\int_R f\,dA$

Today, we'll see how to do computations w/ vectors.

In cartesian coordinates,



at every point there are two direction vectors.

$\vec{e}_x(x,y) = (1,0)$ ($\hat{\imath}$ or $\vec{\imath}$)

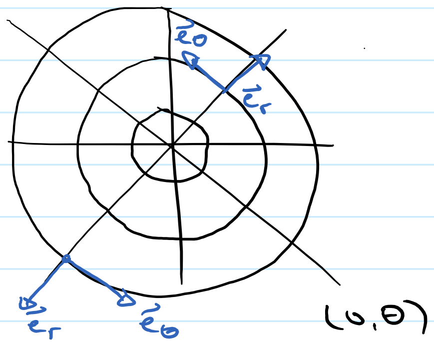
$\vec{e}_y(x,y) = (0,1)$ ($\hat{\jmath}$ or $\vec{\jmath}$)

Can think of these as constant and mutually orthogonal vector fields.

If $\vec{r}(t) = x(t)\vec{e}_x + y(t)\vec{e}_y$

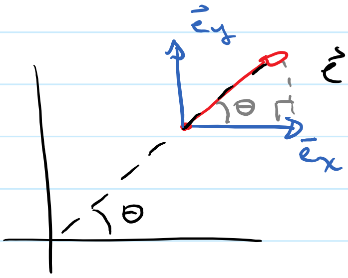
$$\frac{d\vec{r}(t)}{dt} = x'(t)\vec{e}_x + y'(t)\vec{e}_y$$

The analogous direction vector fields in polar coordinates are no longer constant.

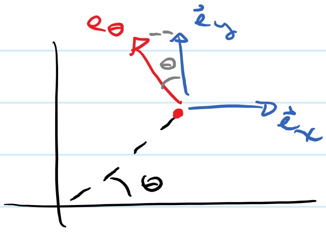


Other names:

$$\left.\begin{aligned}\vec{e}_r &= \hat{r} = \vec{r}\\ \vec{e}_\theta &= \hat{\theta} = \vec{\theta}\end{aligned}\right]$$



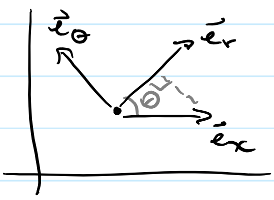
$$\vec{e}_r(x,y) = \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y$$



$$\vec{e}_\theta = -\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_y$$

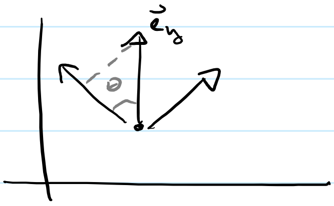
Warning:

$$(r,\theta) \neq r\vec{e}_r + \theta\vec{e}_\theta$$
$$(x,y) = x\vec{e}_x + y\vec{e}_y$$
$$(r,\theta) = r\vec{e}_r\,(0,\theta)$$

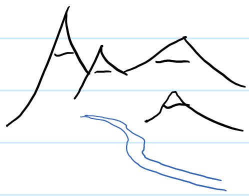


$$\vec{e}_r\cdot\vec{e}_\theta = (\cos\theta\vec{e}_x + \sin\theta\vec{e}_y)\cdot(-\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_y)$$
$$= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

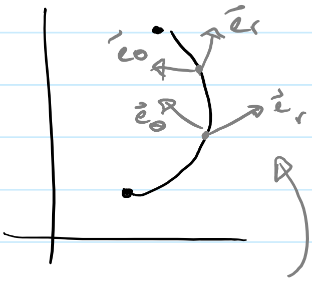
$$\vec{e}_x = \cos\theta\,\vec{e}_r - \sin\theta\,\vec{e}_\theta$$



$$\vec{e}_y = \sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta$$

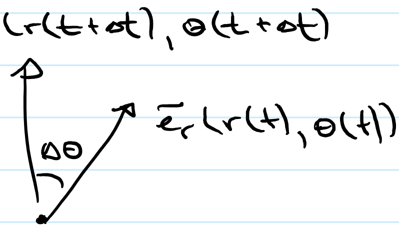


Let $t \to \vec{r}(t)$, $t \in [a,b]$



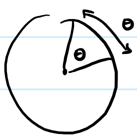
* these vectors should probably be more parallel to eachother...

$$\frac{d\vec{e}_r}{dt}(r(t),\theta(t)) = \frac{d\theta}{dt}\vec{e}_\theta$$

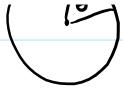


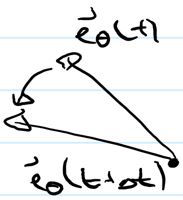
where $\Delta\theta = \theta(t+\Delta t) - \theta(t)$

$$\frac{\vec{e}_r(t+\Delta t) - \vec{e}_r(t)}{\Delta t} \approx \frac{\Delta\theta}{\Delta t}\vec{e}_\theta$$



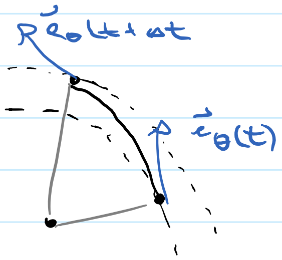
$\vec{e}_\theta(t)$





$$\frac{\vec{e}_\theta(t+\Delta t)-\vec{e}_\theta(t)}{\Delta t} \approx -\frac{\Delta r}{\Delta t}\vec{e}_r$$

* Check correction next lecture notes.



$$\frac{d\vec{e}_\theta}{dt}(r(t),\theta(t)) = -\frac{dr}{dt}\vec{e}_r(r(t),\theta(t))$$

$$\vec{r}(t) = r(t)\,\vec{e}_r(0,\theta(t))$$
$$\vec{v}(t) = \frac{d\vec{r}}{dt}(t) = \frac{dr}{dt}\vec{e}_r(0,\theta(t)) + r(t)\frac{d\vec{e}_r}{dt}(0,\theta(t))$$
$$= \frac{dr}{dt}\vec{e}_r + r\left(\frac{d\theta}{dt}\vec{e}_\theta\right)$$
$$= \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$

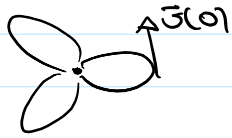
$$\|\vec{v}(t)\| = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}$$

$$\vec{a}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{e}_r + \left(r\frac{d^2\theta}{dt^2} - 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\vec{e}_\theta$$

$$\vec{F}(r,\theta) = F_r(r,\theta)\,\vec{e}_r(r,\theta) + F_\theta(r,\theta)\,\vec{e}_\theta(r,\theta)$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b F_r(r(t),\theta(t))\frac{dr}{dt} + F_\theta(r(t),\theta(t))\,r\frac{d\theta}{dt}\,dt$$

<u>Example</u>:

$$(r(t),\theta(t)) = (1+\cos(3t),\,t)$$



$$\frac{dr}{dt} = -3\sin(3t)$$
$$\frac{d\theta}{dt} = 1$$

$$\vec{r}(t) = (-3\sin(3t))\vec{e}_r + (1+\cos(3t))\cdot 1\,\vec{e}_\theta$$

$t=0$: $0\vec{e}_r + 2\vec{e}_\theta$

$t=\pi/3$: $0\vec{e}_r + 0\vec{e}_\theta$

$t=\pi/6$: $-3\vec{e}_r + 1\vec{e}_\theta$