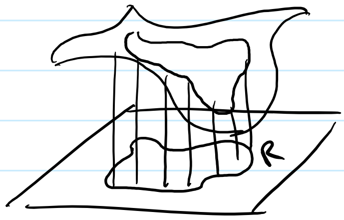
L15: Fubini's Theorem

October 17, 2016 12:31 PM

<u>Today</u>: Fubini's Theorem for more general regions.

<u>Reminder</u>: $\iint_R f(x,y)\,dA$

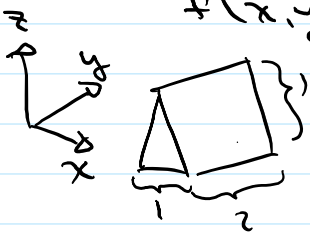
$$\lim_{\text{Area}(R_{ij})\to 0} \sum_i \sum_j f(x_{ij}, y_{ij})$$



Area(R_{ij})

<u>Example</u>:

$$f(x,y) = x \qquad R = [0,1]\times[0,2]$$

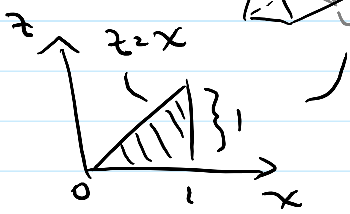


$$\left(\frac{1\cdot 1}{2}\right)\cdot 2 = 1$$
$$= \iint_R f\,dA$$

Computing this integral using iterated integrals

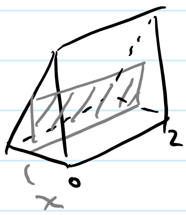
- Fixed y

Let $A(y)$ be the area of the triangular slice w/ y-coordinate $= y$

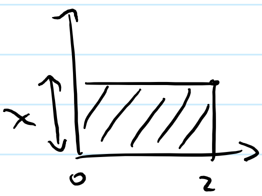


$$A(y) = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$
$$\iint_R f\,dA = \int_0^2 A(y)\,dy = \int_0^2 \frac{1}{2}\,dy = 1$$
$$\parallel$$
$$\int_0^2 \left(\int_0^1 x\,dx\right) dy$$

- Fixed x



Let $A(x)$ be the area of the rectangular slice w/ x-coordinate $= x$



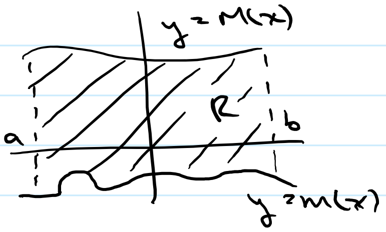
$$A(x) = \int_0^2 x\,dy = 2x$$
$$\iint_R f\,dA = \int_0^1 A(x)\,dx = \int_0^1 2x\,dx$$
$$= \left[x^2\right]_0^1 = 1$$
$$\parallel$$
$$\int_0^1 \left(\int_0^2 x\,dy\right) dx$$

Def^n

Defn

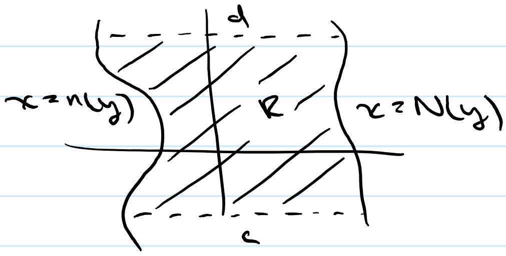
A region R in $\mathbb{R}^2$ is said to be of Type I if there exists functions $m(x)$, $M(x)$, such that

$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} a \le x \le b \\ m(x) \le y \le M(x) \end{array} \right\}$$

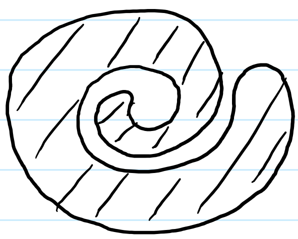


Type II: there exists $n(y)$, $N(y)$ so that:

$$R = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} n(y) \le x \le N(y) \\ c \le y \le d \end{array} \right\}$$



Type III: Both Type I and Type II



– Neither Type I or Type II

Theorem (Fubini)

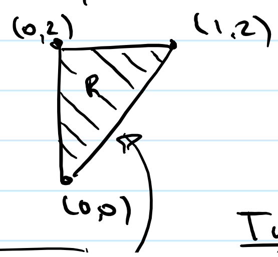
- If R is Type I,

$$\iint_R f\,dA = \int_a^b \left(\int_{m(x)}^{M(x)} f(x,y)\,dy \right) dx$$

- If R is Type II,

$$\iint_R f\,dA = \int_c^d \left(\int_{n(y)}^{N(y)} f(x,y)\,dx \right) dy$$

Example



is Type III

Type I: | Type II:

b. (0,0)

$\boxed{\begin{matrix} y = 2x \\ x = y/2 \end{matrix}}$

Type I: $\left\{(x,y) : \begin{matrix} 0 \le x \le 1 \\ 2x \le y \le 2 \end{matrix}\right\}$

Type II: $\left\{(x,y) : \begin{matrix} 0 \le x \le y/2 \\ 0 \le y \le 2 \end{matrix}\right\}$

Let $f(x,y) = xy$ — random function

Type I: $$\iint_R f\,dA \underset{\uparrow \text{ Fubini}}{=} \int_0^1 \left(\int_{2x}^2 xy\,dy \right) dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} \right]_{2x}^2 dx$$

$$= \int_0^1 \frac{4x}{2} - \frac{4x^3}{2}\,dx$$

$$= \int_0^1 2x - 2x^3\,dx$$

$$= \left[x^2 - \tfrac{1}{2}x^4 \right]_0^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Type II: $$\iint_R f\,dA = \int_0^2 \left(\int_0^{y/2} xy\,dx \right) dy$$

$$= \int_0^2 \left[\frac{x^2 y}{2} \right]_0^{y/2} dy$$

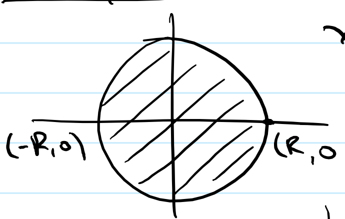
$$= \int_0^2 \frac{y^3}{8}\,dy = \left[\frac{y^4}{32} \right]_0^2$$

$$= \frac{16}{32} = \frac{1}{2}$$

same! :)

__Example__

$x^2 + y^2 = R^2$ is Type III



$y^2 = R^2 - x^2$

$y = \pm\sqrt{R^2 - x^2}$

Type I: $\left\{(x,y) : \begin{matrix} -R \le x \le R \\ -\sqrt{R^2 - x^2} \le y \le \sqrt{R^2 - x^2} \end{matrix}\right\}$

Type II: $\left\{(x,y) : \begin{matrix} \sqrt{R^2 - x^2} \le x \le \sqrt{R^2 - x^2} \\ -R \le y \le R \end{matrix}\right\}$

$f(x,y) = 1$

$$\int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} dy\,dx = \int_{-R}^{R} 2\sqrt{R^2 - x^2}\,dx$$

is confusing subs... don't need to know how to do.

$x = -R\cos\theta$

$dx = R\sin\theta\,d\theta$

$$= 2\sqrt{R^2 - R^2\cos^2\theta}\,R\sin\theta\,d\theta$$

Graph of $-R\cos\theta$

$$dx = R\sin\theta \, d\theta$$

$$= \int_0^{\pi} 2\sqrt{R^2 - R^2\cos^2\theta}\; R\sin\theta \, d\theta$$

$$= \int_0^{\pi} 2R^2\sqrt{\sin^2\theta}\, \sin\theta \, d\theta$$

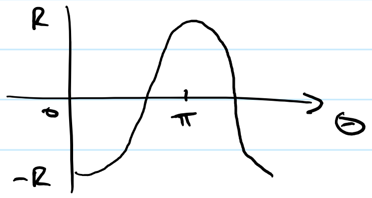
$$= 2R^2 \int_0^{\pi} \sin^2\theta \, d\theta$$

$$= 2R^2 \int_0^{\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= 2R^2 \left(\frac{\pi}{2} - \frac{\sin 2\theta}{4} \Big|_0^{\pi} \right)$$

$$= \pi R^2$$

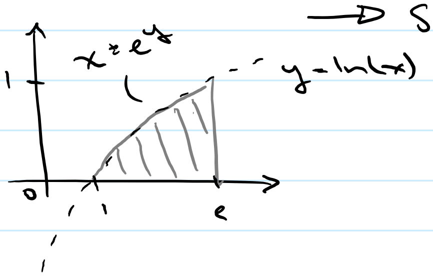
Graph of $-R\cos\theta$



Example

$$\int_1^e \int_0^{\ln(x)} dy\,dx$$

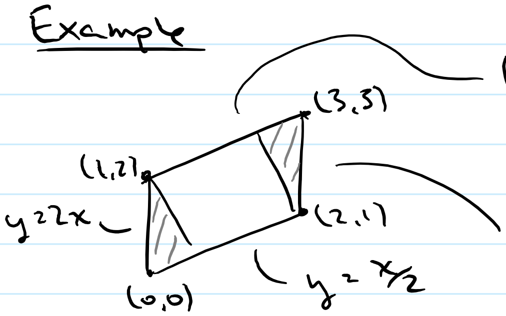
$\Rightarrow$ switch the order of integration.



$$\int_0^1 \int_{e^y}^{e} dx\,dy$$

$$= \int_0^1 e - e^y \, dy \Rightarrow \text{simpler than } \int_1^e \ln(x)\,dx$$

Example



$$(y-2) = \frac{3-2}{3-1}(x-1)$$

$$y = \frac{x}{2} + \frac{3}{2}$$

$$(y - y_0) = m(x - x_0)$$

$$y = 2x + c$$

$$c = y - 2x$$

$$= 1 - 4$$

$$= -3$$

$$y = 2x - 3$$

$$\iint_R f\,dA = \int_0^1 \int_{x/2}^{2x} f(x,y)\,dy\,dx + \int_1^2 \int_{x/2}^{x/2 + 3/2} f(x,y)\,dy\,dx + \int_2^3 \int_{2x-3}^{x/2+3/2} f(x,y)\,dy\,dx$$