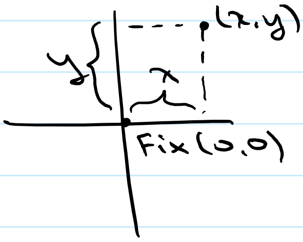
L14: Polar Coordinates

October 13, 2016 1:30 PM

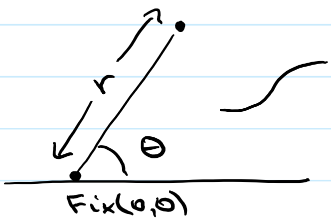
Today: Polar Coordinates

Q:
How to describe the location of a point in $\mathbb{R}^2$?

One A: Cartesian Coordinates



Another A:



A pair (r,θ) is called the polar coordinates of the point $\vec{r}$.

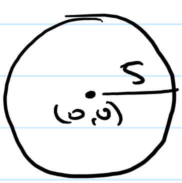
Just like Cartesian coordinates,

$$f(r,\theta) = 0$$

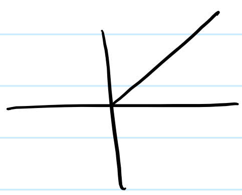
defines a curve in $\mathbb{R}^2$

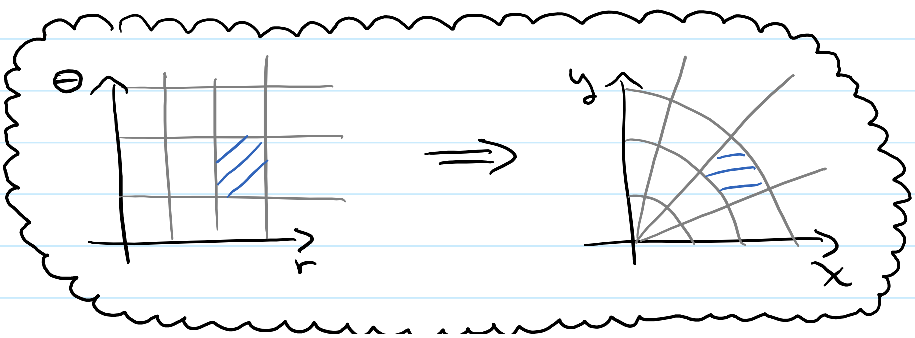
Examples

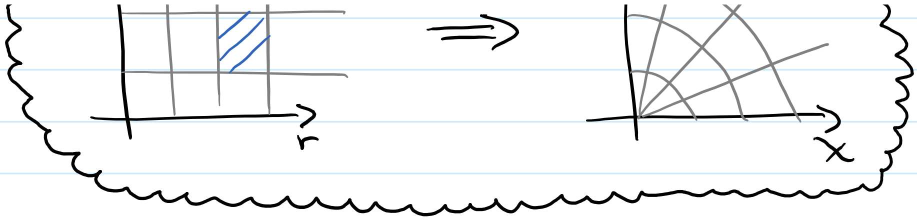
$r = 5$



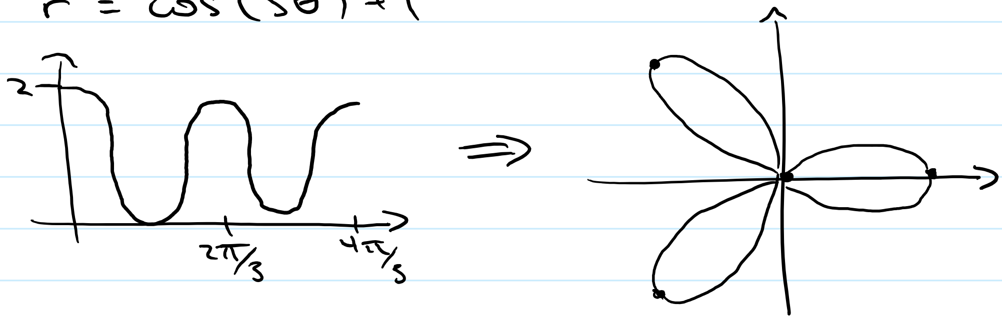
$\theta = \pi/4$





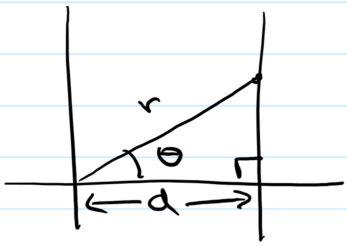


$r = \cos(3\theta) + 1$



$r = \theta$

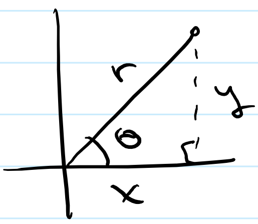
Archemedian spiral
(Anything of the form $r = a + b\theta$ looks similar)



$\frac{r}{d} = \sec\theta$
$\hookrightarrow$ so the eq^n of this line is:
$r = d\sec\theta$

Warning: Many (r,θ) can give the same point.
eg. $(1, \pi/4) = (1, 2\pi + \pi/4) = \ldots$
(r, θ)

Converting between polar and Cartesian

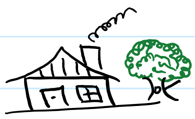


$x = r\cos\theta$
$y = r\sin\theta$

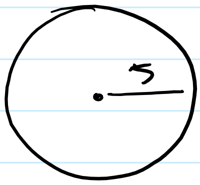
$r = \sqrt{x^2 + y^2}$
$\theta = \arctan\left(\frac{y}{x}\right)$
(with a few special cases)

Parametrized paths

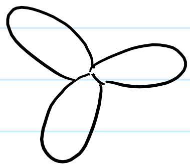
$t \to (r(t), \theta(t))$



Example

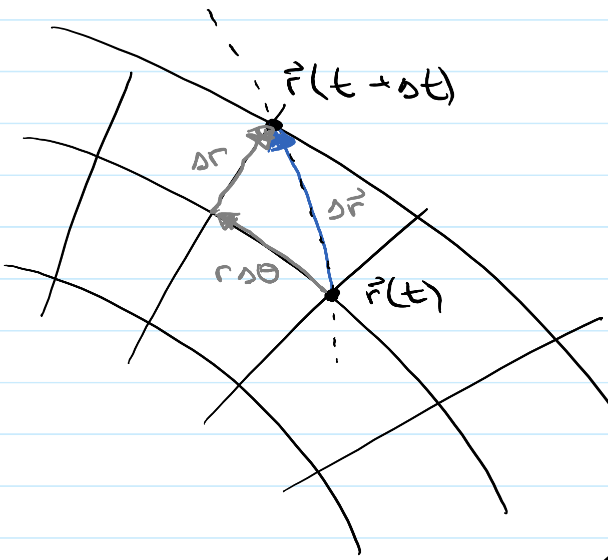


$$t \to (5, t)$$



$$t \to (\cos(3t)+1, t)$$

Integrels of real-valued functions along paths



Expect:

$$\|\Delta \vec r\| \simeq \sqrt{(\Delta r)^2 + (r\Delta\theta)^2}$$

$$\left\|\frac{\Delta \vec r}{\Delta t}\right\| \simeq \sqrt{\left(\frac{\Delta r}{\Delta t}\right)^2 + r^2\left(\frac{\Delta\theta}{\Delta t}\right)^2}$$

$$\xrightarrow{\Delta t \to 0} \|\vec v(t)\| \simeq \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}$$

Let's show this rigorously:

$$\|\vec v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$x(t) = r(t)\cos(\theta(t))$$

$$y(t) = r(t)\sin(\theta(t))$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt}$$

$$= \cos\theta \, {dr}/{dt} - r\sin\theta \, {d\theta}/{dt}$$

$$\frac{dy}{dt} = \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}$$

$$\left(\frac{dx}{dt}\right)^2 = \cos^2\theta\left(\frac{dr}{dt}\right)^2 - 2r\cos\theta\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} + r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2$$

$$\left(\frac{dy}{dt}\right)^2 = \sin^2\theta\left(\frac{dr}{dt}\right)^2 + 2r\cos\theta\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} + r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

Example

$t \to (5, t) \qquad t \in [0, 2\pi]$

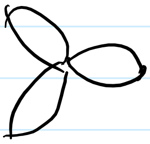
$\frac{dr}{dt} = 0$

$\frac{d\theta}{dt} = 1$

Arclength

$$\int_0^{2\pi} \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{5^2 \cdot 1}\, dt = 10\pi$$



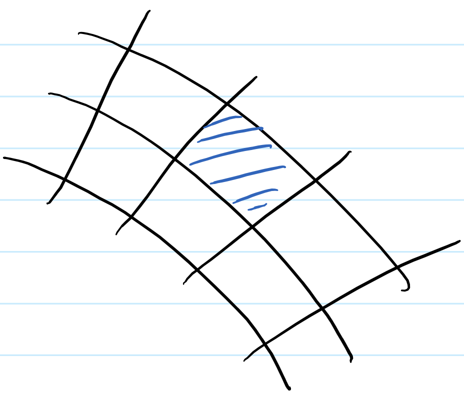
$t \to (1+\cos(3t), t) \qquad t \in [0, 2\pi]$

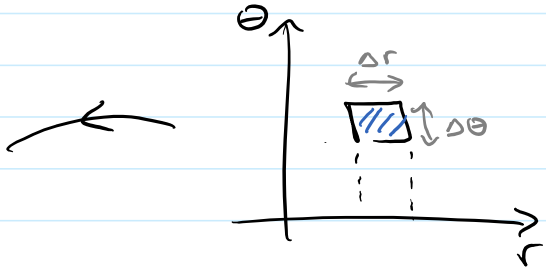
$\frac{dr}{dt} = -3\cos(3t)$

$\frac{d\theta}{dt} = 1$

$$\int_0^{2\pi} \sqrt{(-3\sin(3t))^2 + (1+\cos(3t))^2 \cdot 1}\, dt$$

Can't find an elementary antiderivative.



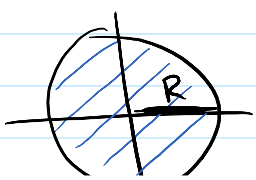


Area $\approx r\Delta\theta\Delta r$

Double integral over $[a,b] \times [c,d]$ in polar coords can be evaluated as:

$$\int_c^d \left(\int_a^b f(r,\theta)\, r\, dr\right) d\theta$$

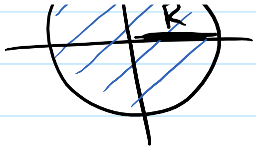
$$= \int_a^b \left(\int_c^d f(r,\theta)\, r\, d\theta\right) dr$$



$0 \le \theta \le 2\pi$

$0 \le r \le R \qquad = [0,R] \times [0, 2\pi]$

$\int_0^{2\pi} \Big(\int_0^R$ ...



$0 \leq r \leq R$ $\quad = [0,R] \times [0,2\pi]$

$$\int_0^{2\pi} \left(\int_0^R r\,dr \right) d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} \right]_0^R d\theta$$

$$= \int_0^{2\pi} \frac{R^2}{2}\, d\theta = \frac{R^2}{2} \left[\theta \right]_0^{2\pi}$$

$$= \frac{2\pi R^2}{2} = \pi R^2$$