L13: Double Integrals

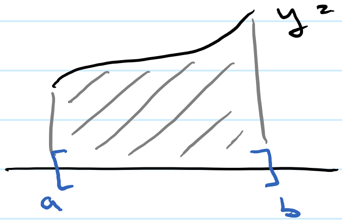
October 12, 2016 11:30 AM

Today: Double Integrals

Given a region R in $\mathbb{R}^2$ and a real-valued function f on R, want to define:

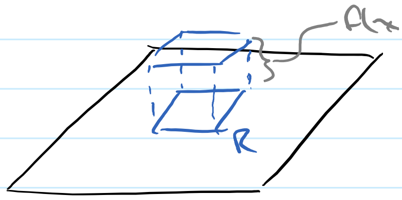
$$\iint_R f\, dA = \text{signed volume bounded by the graph of } f$$

Single variable case:

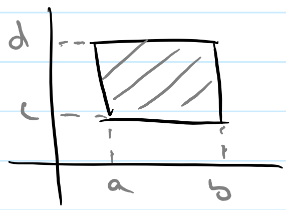


To make this precise, approximate by simple functions and take the limit.

Local Picture:



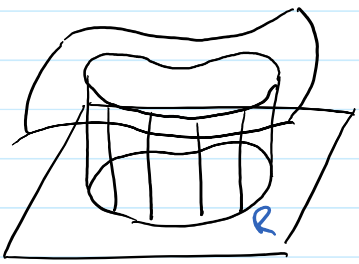
If $R = [a,b] \times [c,d]$
$= \{ (x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d \}$

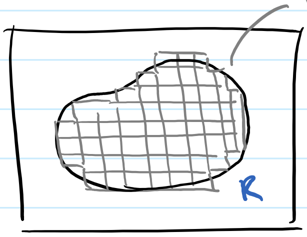


and $f(x,y) = c$, a constant function

$$\iint_R f\, dA = c \cdot \text{Area}(R)$$

Global Picture:





Approximate R by a mesh of rectangles R_{ij} and approximate f by

$f(x_{ij}, y_{ij})$ where $(x_{ij}, y_{ij}) \in R_{ij}$

$$\iint_R f\,dA \approx \sum_i \sum_j f(x_{ij}, y_{ij}) \cdot \text{Area}(R_{ij})$$

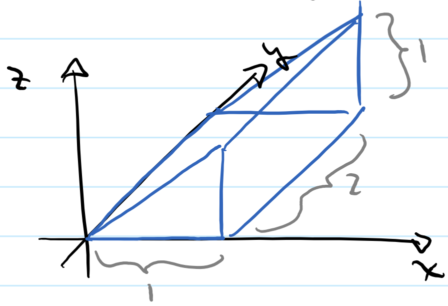
should be

Def^n:

$$\iint_R f\,dA := \lim_{\text{Area}(R_{ij}) \to 0} \sum_i \sum_j f(x_{ij}, y_{ij})\,\text{Area}(R_{ij})$$

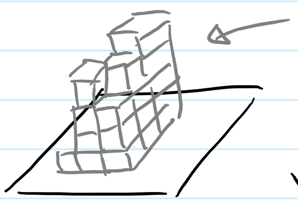
Example:

$f(x,y) = x \qquad R = [0,1] \times [0,2]$



$$\iint_R f\,dA = \left(\frac{1 \cdot 1}{2}\right) \cdot 2$$
$$= 1$$

How to compute?



Volume = ?

At least three ways to compute

1. Count the number of blocks.
2. Over each block in the base rectangle, count the number of blocks above it, and then add results.

   Analogue of $\iint_R f\,dA$

3. Count # of blocks above each rectangle, then add the results over each column, then add the results over columns. Every time you tally in a column, you are computing

   $\int_{\text{column}} f(x,y)\,dy$, with x fixed.

   Then, you add up all of these.

   $$\int\left(\int_{\text{column}} f(x,y)\,dy\right)dx = \iint_R f\,dA$$

4. $$\int\left(\int_{\text{row}} f(x,y)\,dx\right)dy = \iint_R f\,dA$$

* same as 3 but w/ rows

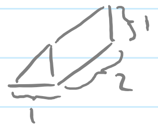
Theorem (Fubini over Rectangles)

If $R = [a,b] \times [c,d]$

$$\iint_R f\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$
$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Example

$f(x,y) = x$ over $R = [0,1] \times [0,2]$



$\left(\frac{1 \cdot 1}{2} \right) \cdot 2 = 1$

$$\int_0^1 \left(\int_0^2 x\,dy \right) dx$$
$$= \int_0^1 [xy]_{y=0}^{y=2}\,dx = \int_0^1 (2x - 0)\,dx$$
$$= [x^2]_{x=0}^{x=1} = 1^2 - 0^2$$
$$= 1$$

$$\int_0^2 \left(\int_0^1 x\,dx \right) dy = \int_0^2 \left[\frac{x^2}{2} \right]_{x=0}^{x=1} dy$$
$$= \int_0^2 \left(\frac{1^2}{2} - \frac{0^2}{2} \right) dy$$
$$= \int_0^2 \frac{1}{2}\,dy = \left[\frac{y}{2} \right]_{y=0}^{y=2}$$
$$= \frac{2-0}{2}$$
$$= 1$$

$\iint_R y \sin(xy)\,dA \qquad R = [1,2] \times [0,\pi]$

$$\int_0^{\pi} \left(\int_1^2 y \sin(xy)\,dx \right) dy$$
$$= \int_0^{\pi} [-\cos(xy)]_{x=1}^{x=2}\,dy$$
$$= \int_0^{\pi} -\cos(2y) + \cos(y)\,dy$$
$$= \left[-\frac{\sin(2y)}{2} + \sin(y) \right]_{y=0}^{y=\pi} = 0$$

Integrating w/r/t y first, need to do integration by parts twice.

so basically a bad idea...

Some general properties:

(i) $\iint_R f+g \, dA = \iint_R f \, dA + \iint_R g \, dA$

(ii) $\iint_R cf \, dA = c \iint_R f \, dA$, $c \in \mathbb{R}$

(iii) If $f(x,y) \leq g(x,y)$ for all $(x,y) \in R$

$\iint_R f \, dA \leq \iint_R g \, dA$

(iv) $\left| \iint f \, dA \right| \leq \iint |f| \, dA$