L12: Flux across a curve

October 6, 2016 1:30 PM

Midterm: Oct 25th 6pm–8pm (Stir. A)

Today: <u>Flux across a curve.</u>

<u>Local Picture</u>:

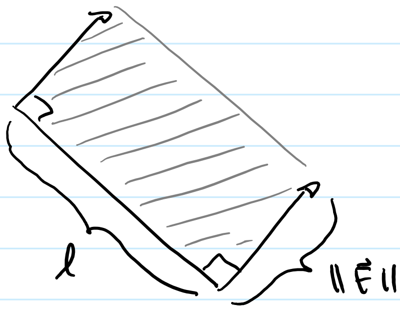
Suppose a fluid flows with constant velocity $\vec{F}$ on $\mathbb{R}^2$

<u>Q</u>:

How much fluid flows across a line segment of length l per unit time?

• Perpendicular case:

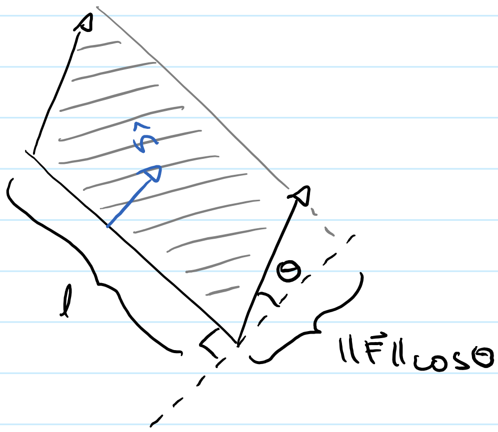
<u>A</u>: $\|\vec{F}\| \cdot l$



• Skew case:

answer

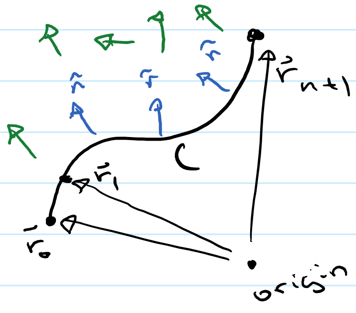
<u>A</u>: $\|\vec{F}\| \cos\theta \cdot l$



if we write a unit normal to l as $\hat{n}$:

$$\vec{F} \cdot \hat{n} \cdot l$$

<u>Global picture</u>:



Same question for a curve C, with a choice of normal direction.

For $\vec{r}_{i+1} - \vec{r}_i$, the flow across is approximately

$$\vec{F}(\vec{r}_i) \cdot \hat{n}(\vec{r}_i) \| \vec{r}_{i+1} - \vec{r}_i \|$$

So, the total flow:

$$\sum \vec{F}(\vec{r}_i)\cdot\hat{n}(\vec{r}_i)\,\|\vec{r}_{i+1}-\vec{r}_i\|$$

Parametrize C as $t \mapsto \vec{r}(t)$, $t \in [a,b]$

Choose t_i so that $\vec{r}(t_i) = \vec{r}_i$

$$\sum \vec{F}(\vec{r}_i)\cdot\hat{n}(\vec{r}_i)\,\|\vec{r}_{i+1}-\vec{r}_i\|$$

$$\approx \sum \vec{F}(\vec{r}(t_i))\cdot\hat{n}(\vec{r}(t_i))\,\|\vec{v}(t_i)\|(t_{i+1}-t_i)$$

treating velocity as constant on each $\vec{r}_{i+1}-\vec{r}_i$

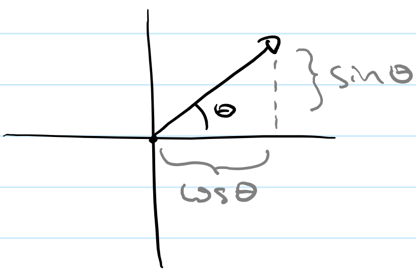
$$\xrightarrow[\Delta\vec{r}\to 0]{} \int_a^b \vec{F}(\vec{r}(t))\cdot\hat{n}(r(t))\,\|\vec{v}(t)\|\,dt$$

Called the flux of $\vec{F}$ across C in direction $\hat{n}$.

$$\int_C \vec{F}\cdot\hat{n}\,ds$$

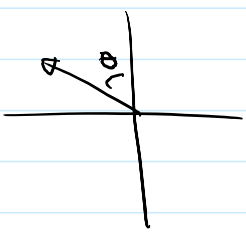
Rotating vectors in $\mathbb{R}^2$

Q:
If the vector $(1,0)$ $(=\vec{i})$ is rotated θ radians counterclockwise, what are the components of the resulting vector?



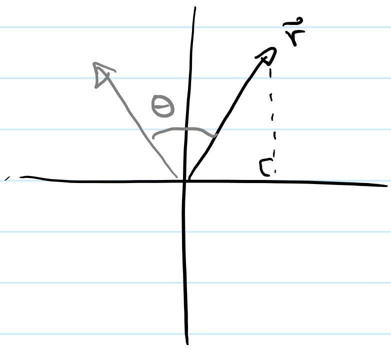
A: $(\cos\theta, \sin\theta)$

Q:
Same question for $(0,1)$ $(=\vec{j})$



A: $(-\sin\theta, \cos\theta)$

Q:
Same question for $\vec{r} = (x,y)$



A: $x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta)$
$= (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

$$\left[\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right]$$

rotation matrix

Special cases:

$\theta = \pi/2$ rad, $(x,y) \mapsto (x\cos\pi/2 - y\sin\pi/2,\ x\sin\pi/2 + y\cos\pi/2)$
$= (-y, x)$

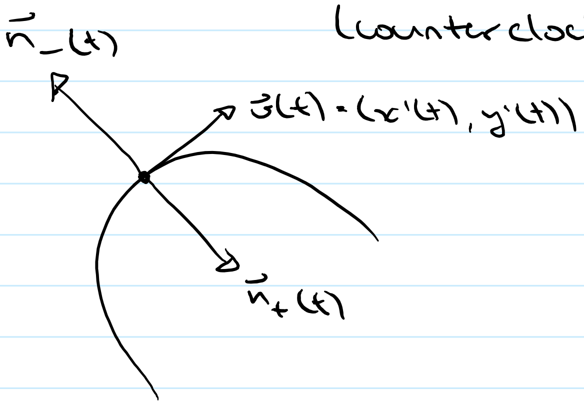
$\theta = -\pi/2$ rad, $(x,y) \mapsto (y, -x)$

This gives a way of finding normals.

If a curve C is parametrized by $t \mapsto \vec{r}(t)$, $(t \in [a,b])$
Define the clockwise and counterclockwise normals to the parametrization of C as

$t \mapsto \vec{n}_+(t) := (y'(t), -x(t))$
(clockwise)

$\vec{n}_-(t) := (-y'(t), x'(t))$
(counterclockwise)



Remark

$\vec{n}_+(t) = \hat{n}_+(t)\|\vec{v}(t)\|$

(same for $\vec{n}_-$)

Conclusion

To compute flux, find:

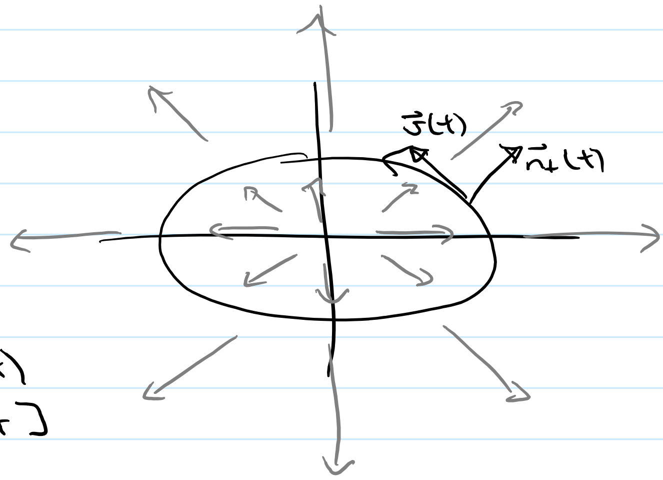
$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}_+(t)\,dt$ or

$$\int_a \vec F(\vec r(t)) \cdot \vec n_+(t)\,dt \quad \text{or}$$
$$\int_a^b \vec F(\vec r(t)) \cdot \vec n_-(t)\,dt$$

Example:

$\vec F(x,y) = (x,y)$

across an ellipse centered at the origin, oriented outward



$t \mapsto (A\cos t, B\sin t) =: \vec r(t)$

$t \in [0, 2\pi]$

$\vec v(t) = (-A\sin t,\ B\cos t)$

$\vec n_+(t) = (B\cos t,\ A\sin t)$

$$\begin{aligned}\vec F(\vec r(t)) \cdot \vec n_+(t) &= (A\cos t, B\sin t) \cdot (B\cos t, A\sin t)\\ &= AB\cos^2 t + AB\sin^2 t\\ &= AB(\cos^2 t + \sin^2 t) = AB\end{aligned}$$

$$\text{Flux} = \int_0^{2\pi} AB\,dt = 2\pi AB$$
$$(= 2 \cdot \text{Area}(\text{Ellipse}))$$

Exercise:

$$\vec F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

Then the flux across any circle centered at the origin (oriented outward) is equal.