L11: Finding potential functions

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Last time

Gradient fields ($\vec{\nabla} f$ for some f) are path-independent.

Conversely, path-independent fields are gradient fields.

Finding potential functions

Method 2 in $\mathbb{R}^3$:

$$\vec{F}(x,y,z) = (2x + 2y^2 + 2z^3,\ 4xy + 4y^3 + 4yz^3,\ 6xz^2 + 6y^2z^2 + 6z^5)$$

If a partial function f exists,

$$\frac{\partial f}{\partial x} = F_1(x,y,z) = 2x + 2y^2 + 2z^3$$

Integrate with respect to x

$$f(x,y,z) = x^2 + 2xy^2 + 2xz^3 + \overbrace{g(y,z)}^{\text{"}+C\text{" but can depend on } y,z}$$

Take partial with respect to y

$$\frac{\partial f}{\partial y} = 4xy + \frac{\partial g}{\partial y}(y,z)$$

$$= F_2(x,y,z) = 4xy + 4y^3 + 4yz^3$$

$$\frac{\partial g}{\partial y} = 4y^3 + 4yz^3$$

Integrate w/r/t y

$$g(y,z) = y^4 + 2y^2z^3 + h(z)$$

$$f(x,y,z) = x^2 + 2xy^2 + 2xz^3 + y^4 + 2y^2z^3 + h(z)$$

Take partial w/r/t z

$$\frac{\partial f}{\partial z} = 6xz^2 + 6y^2z^2 + h'(z)$$

$$= F_3(x,y,z) = 6xz^2 + 6y^2z^2 + 6z^5$$

$$h'(z) = 6z^5 \Rightarrow h(z) = z^6 + C$$

$$f(x,y,z) = x^2 + y^4 + z^6 + 2xy^2 + 2y^2z^3 + 2z^3x + C$$

$$= (x + y^2 + z^3)^2 + C$$

$\nabla f \stackrel{?}{=} \vec{F}$

$$\frac{\partial f}{\partial x} = 2(x + y^2 + z^3)\cdot 1$$
$$= 2x + 2y^2 + 2z^3 = F_1 \checkmark$$

$$\frac{\partial f}{\partial y} = 2(x + y^2 + z^3)2y$$
$$= 4xy + 4y^3 + 4yz^3 = F_2 \checkmark$$

$$\frac{\partial f}{\partial z} = 2(x + y^2 + z^3)3z^2$$
$$= 6xz^2 + 6y^2z^2 + 6z^5 = F_3 \checkmark$$

Method 3

Begin by fixing some point, say Q.
For any P, define

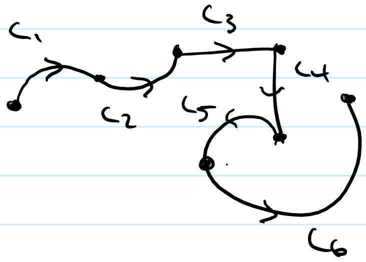
$$f(P) := \int_{C \text{ from } Q \text{ to } P} \vec{F}\cdot d\vec{r}$$

$:=$ ↳ left side defined by the right side

This is well-defined because $\vec{F}$ is assumed path-independent.

Terminology

Sometimes a given path C,



it naturally breaks into finitely many pieces, say $C_1, \ldots, C_n$ that are simpler to parameterize individually.

$$C = C_1 + \ldots + C_n$$

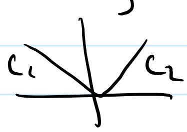
Define,

$$\int_{C_1 + \ldots + C_n} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \ldots + \int_{C_n} \vec{F}\cdot d\vec{r}$$

$$\int_{C_1 + \ldots + C_n} f\,ds = \int_{C} f\,ds + \ldots + \int_{C} f\,ds$$

This also allows working w/ paths like

$$x \longmapsto |x|$$

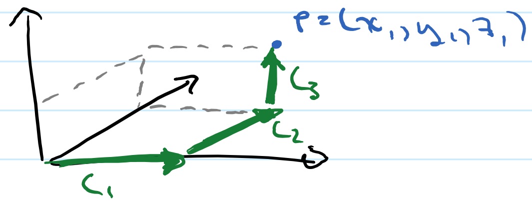


which are differentiable except at finitely many points.

$$\int_{C_1 + \ldots + C_n} \vec{\nabla} f \cdot d\vec{r} = \int_{C_1} \vec{\nabla} f\cdot d\vec{r} + \ldots + \int_{C_n} \vec{\nabla} f\cdot d\vec{r}$$

$$= (f(P_1) - f(Q_1)) + \dots + (f(P_n) - f(Q_n))$$
$$= f(P_1) - f(Q_n)$$

Example of Method 3



$Q = (0,0,0)$

$C = C_1 + C_2 + C_3$, where

$C_1 : t \mapsto (t,0,0)$, t goes from 0 to x_1

$C_2 : t \mapsto (0,t,0)$, t goes from 0 to y_1

$C_3 : t \mapsto (0,0,t)$, t goes from 0 to z_1

The velocities are:

$C_1 : \vec{v}(t) = (1,0,0)$

$C_2 : \vec{v}(t) = (0,1,0)$

$C_3 : \vec{v}(t) = (0,0,1)$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{x_1} F_1(t,0,0)\, dt$$
$$= \int_0^{x_1} (2t + 2\cdot 0^2 + 2\cdot 0^3)\, dt$$
$$= x^2 \Big|_0^{x_1} = x_1^2$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{y_1} F_2(x_1, t, 0)\, dt$$
$$= \int_0^{y_1} (4x_1 t + 4t^3 + 4t\cdot 0^3)\, dt$$
$$= (2x_1 t^2 + t^4)\Big|_0^{y_1} = 2x_1 y_1^2 + y_1^4$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^{z_1} F_3(x_1, y_1, t)\, dt$$
$$= \int_0^{z_1} (6x_1 t^2 + 6y_1^2 t^2 + 6t^5)\, dt$$
$$= (2x_1 t^3 + 2y_1^2 t^3 + t^6)\Big|_0^{z_1}$$
$$= 2x_1 z_1^3 + 2y_1^2 z_1^3 + z_1^6$$

$$f(x,y,z) = \int_C \vec{F} \cdot d\vec{r}$$
$$= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r}$$

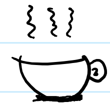
$$= x_1^2 + 2x_1 y_1^2 + y_1^4 + 2x_1 z_1^3 + 2y_1^2 z_1^3 + z_1^6$$

<u>Theorem</u>:

If a vector field is path-independent, then there exists a function f with $\vec{F} = \nabla f$

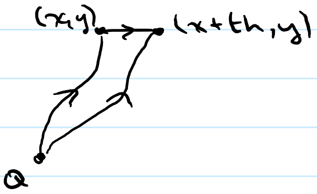
The proof of this theorem justifies Method 3.

Take $f(P) = \int_{C \text{ from some fixed } Q \text{ to } P} \vec{F} \cdot d\vec{r}$



Have to check $\nabla f = \vec{F}$

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+th,y) - f(x,y)}{h}$$



$$\int_{Q \text{ to } (x+th,y)} \vec{F} \cdot d\vec{r} - \int_{Q \text{ to } (x,y)} \vec{F} \cdot d\vec{r}$$

$$= \int_{(x,y) \to (x+th)} \vec{F} \cdot d\vec{r}$$

$t \mapsto (x+th, y) \qquad t \in [0,1]$

$\vec{v}(t) = (h,0)$

$$\int_{\to} \vec{F} \cdot d\vec{r} = \int_0^1 hF(x+th), y)\, dt$$

$$\lim_{h \to 0} \frac{1}{h} \int_0^1 hF_1(x+th,y)\, dt$$

$$\text{"="} \int_0^1 \lim_{h \to 0} F_1(x+th,y)\, dt$$

need further justification to take lim inside.

$$= \int_0^1 F_1(x,y)\, dt = F_1(x,y)$$