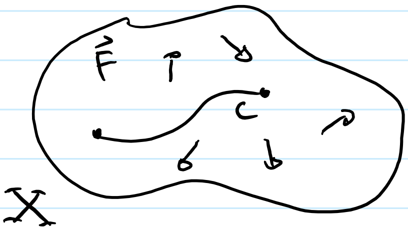
L10: Gradient Fields are Path-Independent

October 3, 2016 12:29 PM

Midterm: Tues Oct. 25^{th} $6^{pm}-8^{pm}$

Today: Gradient fields are path-independent.



Work done by $\vec{F}$ along C

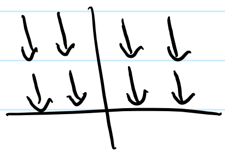
$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{v}(t)\,dt$$

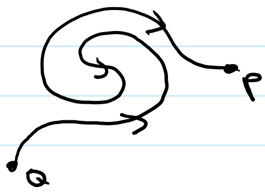
where $t \mapsto \vec{r}(t)$ $t \in [a,b]$ is a parametrization of C

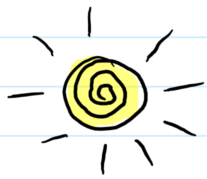
In good cases, $\int_C \vec{F}\cdot d\vec{r}$ only depends on the two endpoints of C.
If this is the case, $\vec{F}$ is called conservative or path-indep.

e.g. $\vec{F} = (0, -mg)$

is conservative field







$$\int_C \vec{F}\cdot d\vec{r} = -mg\,(y(P) - y(Q))$$

Theorem (Fundamental Theorem of Calculus for line integrals)
(path)

Let f be a function defined on $X \subseteq \mathbb{R}^n$ (for us, $n = 1, 2,$ or 3)
Let C be a path contained in X, from Q to P,

$$\int_C \vec{\nabla} f\cdot d\vec{r} = f(P) - f(Q)$$

$$\left(\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\right)$$

Consequence: $\vec{\nabla} f$ is conservative.

Remark: In $\mathbb{R}^1$, $\vec{\nabla} f(x) = \frac{df}{dx}$

Theorem says: $\int_a^b \frac{df}{dx}\cdot\frac{dx}{dt}\,dt$

$$= \int_a^b \frac{d}{dx} f(x(t))\,dt$$

$$= f(b) - f(a)$$

$\uparrow$ Single-Variable Fund. Thm. of Calc.

Proof of Theorem:

Parametrize C as $t \mapsto \vec{r}(t)$, $t \in [a,b]$

$$\int_C \vec{\nabla} f \cdot d\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_a^b \frac{\partial f}{\partial x}(x(t), y(t))\, x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))\, y'(t)\, dt$$

$$= \int_a^b \frac{d}{dt} f(x(t), y(t))\, dt$$

$$= f(x(b), y(b)) - f(x(a), y(a))$$

$$= f(P) - f(Q)$$

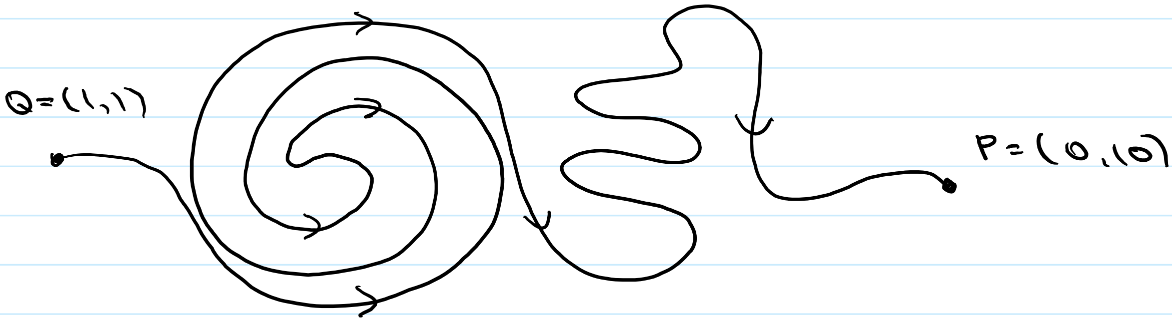
proofed! :)

Example:

Find the work done by

$$\vec{F}(x,y) = (y^2 e^{xy^2}, 2xy e^{xy^2})$$

along the following curve:



Solution:

Let's check that:

$$\vec{F} = \vec{\nabla} f, \text{ where } f = e^{xy^2}$$

$$\frac{\partial f}{\partial x} = e^{xy^2} \cdot \frac{\partial f}{\partial x}(xy^2) = y^2 e^{xy^2}$$

$$\frac{\partial f}{\partial y} = e^{xy^2} \cdot \frac{\partial f}{\partial y}(xy^2) = 2xy e^{xy^2}$$

Therefore,

$\int_C \vec{\nabla F} \cdot d\vec{r}$ for $f = e^{xy^2}$

$$\int_C \vec{F} \cdot d\vec{r} = f(0,10) - f(1,1)$$

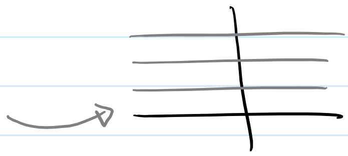
$$= e^{0 \cdot 10^2} - e^{1 \cdot 1^2} = 1 - e$$

Terminology:

- The function f is called a potential for the gradient field $\vec{\nabla} f$

- Level curves of f are also called equipotential curves.

eg. $F(x,y) = (0, -mg)$
$f = -mgy$

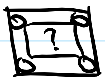


Theorem:
Every conservative vector field is a gradient field.

We'll prove this soon ...

Q:
If we have a vector field that is known to be conservative, how do we find a potential?

Method 1: Guess ...?



Method 2: $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$
Conservative means
$\vec{F} = \vec{\nabla} f$ for some f

In coordinates, this gives two differential equations:

$$F_1(x,y) = \frac{\partial f}{\partial x}(x,y)$$

$$F_2(x,y) = \frac{\partial f}{\partial y}(x,y)$$

Example:

$$\vec{F}(x,y) = (6x + 12xy^2, 12x^2y)$$

$$\frac{\partial f}{\partial x} = 6x + 12xy^2$$

Integrate w/r/t x

$$f \cancel{(x,y)} = 3x^2 + 6x^2y^2 + g(y)$$

Take partial w/r/t y

$$12x^2y = F_2(x,y) = \frac{\partial f}{\partial y} = 12x^2y + g'(y)$$

$$g'(y) = 0 \Rightarrow g(y) = c$$

Arbitrary potential has the form

$$f = 3x^2 + 6x^2y^2 + c$$