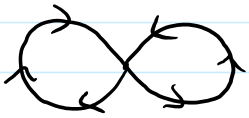
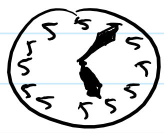
L9: Approximating Work, Path-Independence

September 29, 2016 1:30 PM

Clockwise or Counterclockwise?

$$t \mapsto (2\cos(t), \sin(2t)),\ t \in [0, 2\pi)$$





it's 5 o'clock somewhere!

<u>Terminology</u> about curves:

A parameterized path
$$[a,b] \to \mathbb{R}^2 \text{ or } \mathbb{R}^3$$
$$t \to \vec{r}(t)$$

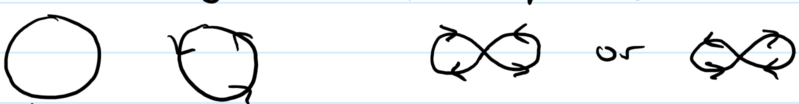
is said to be:

- Simple if it has no self-intersections
  $\vec{r}(t) = \vec{r}(s) \Rightarrow t = s$ (Except if $t=a$ and $s=b$
- Closed if $\vec{r}(a) = \vec{r}(b)$

not simple closed | not simple not closed | simple not closed

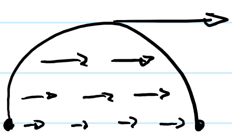
A curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be:

- Parametrizable if there exists a 1-1 parametrization of the curve.
- Oriented if parametrizable and a choice of orientation is made.



not oriented | oriented | both oriented, but different direction

<u>Approximating work</u>:



$$F(x,y) = (y, 0)$$

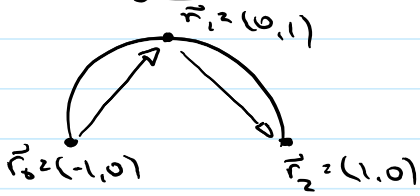
Let's write out a few stages of the limiting sum:

$$\lim_{\Delta r \to 0} \sum \vec{F}(\vec{r}_i) \cdot \Delta \vec{r}_i$$

<u>Stage 1</u>:

$$\vec r_1 - \vec r_0 = (1,0) - (-1,0) = (2,0)$$
$$\vec F(\vec r_0) = 0$$
$$\vec F(\vec r_0)\cdot(\vec r_1 - \vec r_0) = 0 \approx \text{work}$$

Stage 2:

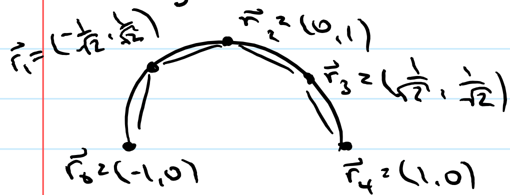


$$\left.\begin{aligned} \vec r_1 - \vec r_0 &= (1,1) \\ \vec r_2 - \vec r_1 &= (1,-1) \end{aligned}\right\} \begin{aligned} \vec F(\vec r_0) &= \vec 0 \\ \vec F(\vec r_1) &= (1,0) \end{aligned}$$

$$\text{Work} \approx 0 + 1 = 1$$

* Skip stage 3

Stage 4:



$$\vec r_1 - \vec r_0 = \left(-\tfrac{1}{\sqrt2} + 1, \tfrac{1}{\sqrt2}\right)$$
$$\vec r_2 - \vec r_1 = \left(\tfrac{1}{\sqrt2}, 1 - \tfrac{1}{\sqrt2}\right)$$
$$\vec r_3 - \vec r_2 = \left(\tfrac{1}{\sqrt2}, \tfrac{1}{\sqrt2} - 1\right)$$
$$\vec r_4 - \vec r_3 = \left(1 - \tfrac{1}{\sqrt2}, -\tfrac{1}{\sqrt2}\right)$$

$$\vec F(\vec r_0) = \vec 0$$
$$\vec F(\vec r_1) = \left(\tfrac{1}{\sqrt2}, 0\right)$$
$$\vec F(\vec r_2) = (1,0)$$
$$\vec F(\vec r_3) = \left(\tfrac{1}{\sqrt2}, 0\right)$$

$$\begin{aligned} \text{work} &\approx 0 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{\sqrt2}\right) + \left(1 - \tfrac{1}{\sqrt2}\right)\cdot\left(\tfrac{1}{\sqrt2}\right) \\ &= 0 + \tfrac{1}{2} + \tfrac{1}{\sqrt2} + \tfrac{1}{\sqrt2} - \tfrac{1}{2} \\ &= 2\tfrac{1}{\sqrt2} = \sqrt2 \approx 1.414 \end{aligned}$$
$$\frac{\pi}{2} \approx \frac{3.14\ldots}{2} \approx 1.57$$

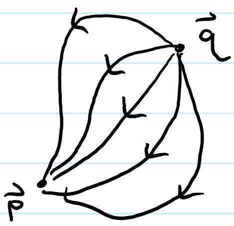
Define the work done by $\vec F$ along C as the limit
$$\lim_{n\to\infty} \sum_{i=0}^{n} F(\vec r_i)\cdot(\vec r_{i+1} - \vec r_i)$$

But, parametrizing C by $t \mapsto \vec r(t)$, and taking t_i so that $\vec r_i = \vec r(t_i)$,
$$\begin{aligned} \vec r_{i+1} - \vec r_i &= r(t_{i+1}) - \vec r(t_i) \\ &\approx \vec v(t_i)(t_{i+1} - t_i) \end{aligned}$$
$$\sum \vec F(\vec r_i)\cdot(\vec r_{i+1} - \vec r_i) \approx \sum \vec F(\vec r(t_i))\cdot \vec v(t_i)(t_{i+1} - t_i)$$
$$\longrightarrow \int_a^b \vec F(\vec r(t))\cdot \vec v(t)\,dt$$

Path-Independence

Def^n

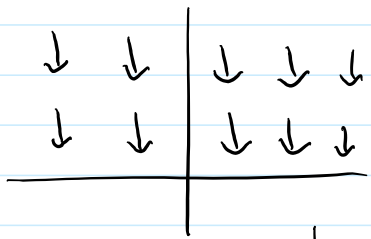
A vector field is said to be conservative (or path independent) if the value of $\int_C \vec{F}\cdot d\vec{r}$ depends only on the endpoints of C (and their order)



If $\vec{F}$ is conservative, work done along all these curves is equal.

Example:

$F(x,y) = (0, -mg)$ Gravity on surface of Earth.

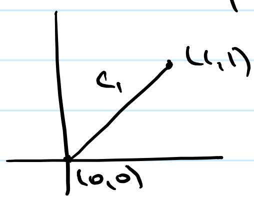


C_1: line segment $t \mapsto (t,t)$ $t \in [0,1]$

$\vec{r}'(t)$

$\vec{v}(t) = (1,1)$

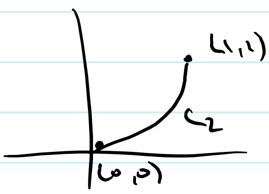
$\vec{F}(\vec{r}(t))\cdot\vec{v}(t) = (-mg)\cdot 1 = -mg$



$$\int_{C_1} \vec{F}\, d\vec{r} = \int_0^1 -mg\, dt$$

$$= -mgt\Big|_0^1 = -mg$$

C_2: parabolic line segment



$t \mapsto (t, t^2)$ $t \in [0,1]$

$\vec{r}'(t)$

$\vec{v}(t) = (1, 2t)$

$\vec{F}(\vec{r}(t))\cdot\vec{v}(t) = -mg\, 2t$

$$\int_{C_2} \vec{F}\, d\vec{r} = \int_0^1 -mg\, 2t\, dt$$

$$= -mg\int_0^1 2t\, dt$$

$$= -mgt^2\Big|_0^1 = -mg(1^2 - 0)$$

$$= -mg$$

C: any parametrized path between $(0,0)$ and $(1,1)$

$t \mapsto \vec{r}(t) = (x(t), y(t))$ $t \in [a,b]$

$\vec{v}(t) = (x'(t), y'(t))$

$\vec{F}(\vec{r}(t)) \cdot \vec{v}(t) = -mg\,y'(t)$

$$\int_a^b -mg\,y'(t)\,dt$$

$$= -mg \int_a^b y'(t)\,dt$$

$$= -mg\,(y(b) - y(a))$$

$$= -mg\,(1-0) = -mg.$$

<u>Goal</u>:

Study necessary and sufficient conditions for a vector field to be conservative.