L8: Directional Derivative, Work

September 28, 2016 11:29 AM

Last time:

Properties of the gradient field $\vec{\nabla} f$

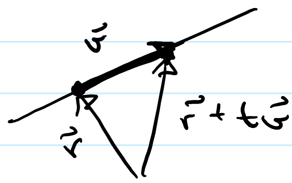
1) ∇f is perpendicular to level curves of f
2) ∇f points in direction of fastest increase of f
3) $\|\nabla f\|$ is proportional to the rate of increase in the direction of fastest increase.

$\rightarrow$ We proved (1) last time.

Def^n:

The directional derivative of f at $\vec{r}$ in direction $\vec{v}$ is the limit:

$$\lim_{t \to 0} \frac{f(\vec{r} + t\vec{v}) - f(\vec{r})}{t}$$



Remark:

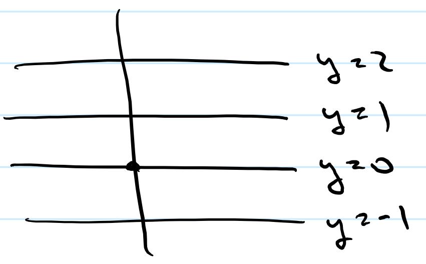
$\frac{\partial f}{\partial x}(\vec{r})$, $\frac{\partial f}{\partial y}(\vec{r})$ are special cases

For $\frac{\partial f}{\partial x}(\vec{r})$, $\vec{v} = (1,0)$

$\frac{\partial f}{\partial y}(\vec{r})$, $\vec{v} = (0,1)$

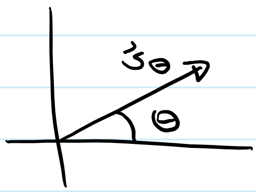
Example

$f(x,y) = y$



$\vec{r} = (0,0)$

$\vec{v}_\theta = (\cos\theta, \sin\theta)$

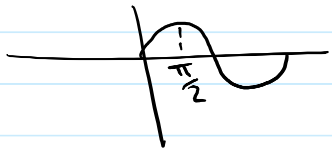


$$\vec{r} + t\vec{v}_\theta = (0,0) + (t\cos\theta, t\sin\theta)$$

$$\frac{f(\vec{r} + t\vec{v}_\theta) - f(\vec{r})}{t} = \frac{t\sin\theta - 0}{t} = \sin\theta$$

So directional derivative of f a $(0,0)$

in direction $\vec{v}_\theta$ is equal to $\sin\theta$



If $g(t) = f(\vec{r} + t\vec{v})$, then

$$\vec{r} + t\vec{v} = (x(t), y(t))$$
$$x(t) = r_1 + tv_1$$
$$y(t) = r_2 + tv_2$$
$$x'(t) = v_1$$
$$y'(t) = v_2$$

$\frac{dg}{dt}(0) =$ directional derivative of f at $\vec{r}$ along $\vec{v}$

If we apply the chain rule, we find

$$\frac{dg}{dt}(0) = \frac{\partial f}{\partial x}(x(0), y(0))x'(0) + \frac{\partial f}{\partial y}(x(0), y(0))y'(0)$$
$$= \frac{\partial f}{\partial x}(\vec{r})v_1 + \frac{\partial f}{\partial y}(\vec{r})v_2$$
$$= \vec{\nabla} f(\vec{r}) \cdot \vec{v}$$

Now we can prove properties (2) and (3)

$$\vec{\nabla} f(\vec{r}) \cdot \vec{v} = \|\vec{\nabla} f(\vec{r})\| \|\vec{v}\| \cos\theta$$

where θ is the angle between $\vec{\nabla} f(\vec{r})$ and $\vec{v}$

The right hand side is maximized when $\theta = 0$
So the directional derivative of f at $\vec{r}$ is maximized when $\vec{v}$ points in the same direction as f.

Conclusion: (2) holds

If we take $\vec{v}$ along $\vec{\nabla} f(\vec{r})$ with $\|\vec{v}\| = 1$

$$\|\vec{\nabla} f(\vec{r})\| = \nabla f(\vec{r}) \cdot \vec{v}$$

= maximal directional derivative

Conclusion: (3) holds

Work:

Let $\vec{F}$ be a vector field defined in $X \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$.

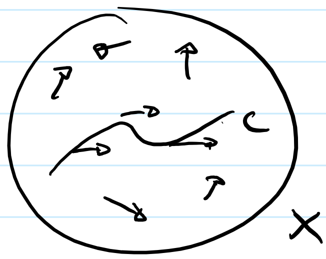
C be a parametrized path contained in X.

$C: t \mapsto \vec{r}(t), \ t \in [a,b]$

Def^n:

The work done by $\vec{F}$ is

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt$$



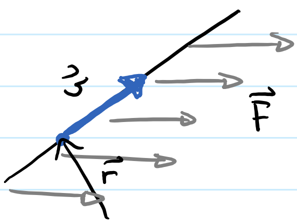
Example:

Suppose $\vec{F}$ is constant

$$\vec{F}(\vec{r}) = \vec{F}$$

C is a straight line

$$t \mapsto \vec{r} + t\vec{v}$$



$\vec{v}(t) = \vec{v}$

$\vec{F}(\vec{r}(t)) \cdot \vec{v}(t) = \vec{F} \cdot \vec{v}$ (independent of t)

$$\int_a^b \vec{F} \cdot \vec{v}\, dt = \vec{F} \cdot \vec{v}(b-a)$$

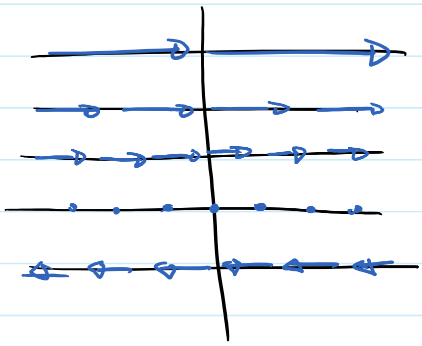
$$= \|\vec{F}\| \|\vec{v}\| \cos\theta\, (b-a)$$

$$= \|\vec{F}\| \cos\theta \cdot \text{Distance}$$

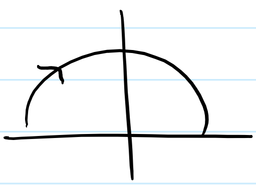
In general, the work depends on C

Example:

$\vec{F}(x,y) = (y, 0)$



C_1: the upper unit semicircle, oriented clockwise



$$t \mapsto (-\cos t, \sin t) = \vec{r}(t)$$

$$t \in [0, \pi]$$

$$\vec{v}(t) = (\sin t, \cos t)$$

$\cos 2t = \cos^2 t + \sin^2 t$

$= (1 - \sin^2 t) - \sin^2 t$

$\vec{F}(\vec{r}(t)) \cdot \vec{v}(t)$

$= (\sin t, 0) \cdot (\sin t, \cos t) = \sin^2 t$

$\int^{\pi}$

$$\left\{ \sin^2 t = \frac{1-\cos(2t)}{2} \right\}$$

$$\int_0 \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt$$

$$= \int_0^{\pi} \sin^2 t\, dt > 0$$

$$= \int_0^{\pi} \frac{1-\cos(2t)}{2}\, dt$$

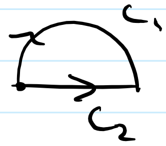
$$= \int_0^{\pi} \frac{1}{2} dt - \int_0^{\pi} \frac{\cos(2t)}{2} dt$$

$$= \frac{\pi}{2} - \frac{1}{4} \underbrace{\sin(2t)\Big|_0^{\pi}}_{0}$$

$$= \frac{\pi}{2}$$

C_2: The line segment

$$t \mapsto \underbrace{(t,0)}_{=\vec{r}(t)} \qquad t \in [-1,1]$$



Notice: $\vec{F}(\vec{r}(t)) = \vec{0}$

$$\vec{F}(\vec{r}(t)) \cdot \vec{v}(t) = 0$$

So $\displaystyle\int_{-1}^{1} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt = 0$

__Dependence on orientation:__

The work is independent of the particular parametrization, as long as it preserves the direction.

__Example:__

C_3: $t \mapsto (\cos t, \sin t) \qquad t \in [0,\pi]$

$\vec{F}(x,y) = (y,0)$

$$\vec{v}(t) = (-\sin t, \cos t)$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{v}(t) = (\sin t, 0) \cdot (-\sin t, \cos t)$$

$$= -\sin^2 t$$

So, $\displaystyle\int_{C_3} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt = -\int_{C_1} \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)\, dt$

This is always true: changing direction changes the sign between ± 1.