L7: Flow Lines, Gradient

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Trick to show Ae^t are the only solutions to $x'(t) = x(t)$:

Look at $\frac{d}{dt}\left(e^{-t}x(t)\right)$

$$= -e^{-t}x(t) + e^{-t}x'(t)$$
$$= -e^{-t}x(t) + e^{-t}x(t)$$
$$= 0$$
$$\Rightarrow e^{-t}x(t) = A$$
$$\Rightarrow x(t) = Ae^{t}$$

Change of notation:

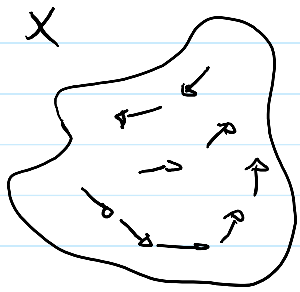
Denote the components of a vector field by

$$(x,y,z) \longmapsto \left(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\right)$$

(instead of F_x, F_y, F_z)

<u>Reminder:</u>

A flow line of a vector field $\vec{F}$ on X is a parameterized path $t \mapsto \vec{r}(t)$ such that $\vec{v}(t) = F(\vec{r}(t))$

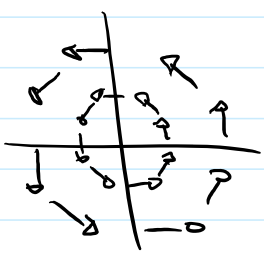


If you drop a seed at some point, where will it go under the flow described by the vector field?

$\rightarrow$ the seed's path is a flow line

Flow lines of the vector field:

$$\vec{F}(x,y) = (-y, x)$$



Guess: Path lines should be circles:

$$t \mapsto \left(A\cos(t+\phi), A\sin(t+\phi)\right)$$

$t \in I$ some time interval

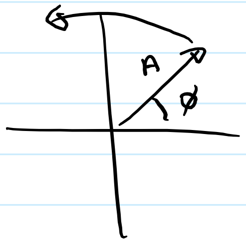
Let's check:

If $\vec{r}(t) = (x(t), y(t))$ is a flow line, $\vec{v}(t) = \vec{F}(\vec{r}(t))$ becomes coordinates

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

$$x'(t) = \frac{d}{dt}\left(A\cos(t+\phi)\right)$$
$$= -A\sin(t+\phi) = -y(t)$$
$$y'(t) = \frac{d}{dt}\left(A\sin(t+\phi)\right)$$
$$= A\cos(t+\phi) = x(t)$$

So any path of this form is a flow line and it turns out, these are all of them.

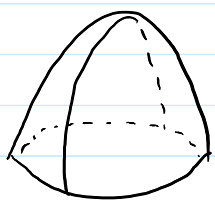


A - amplitude
ϕ - phase shift
set of initial positions is in correspondence with pairs (A, ϕ)

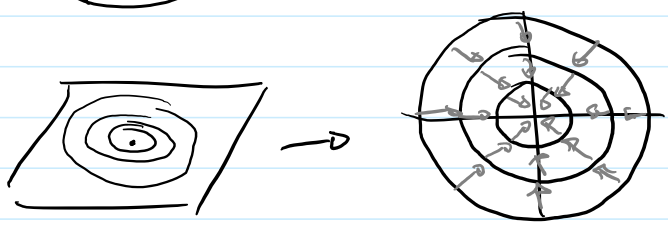
<u>Gradient</u>:
Given a function $f: X \mapsto \mathbb{R}$, $X \subseteq \mathbb{R}^3$, we can define a vector field:

nabla $\rightarrow$ $\vec{\nabla} f : (x,y,z) \mapsto \left(\frac{\partial f}{\partial x}(x,y,z), \frac{\partial f}{\partial y}(x,y,z), \frac{\partial f}{\partial z}(x,y,z)\right)$

<u>Example</u>:
$f = 5 - x^2 - y^2$



Level curves:
$c = 5 - x^2 - y^2$ or $x^2 + y^2 = 5 - c$

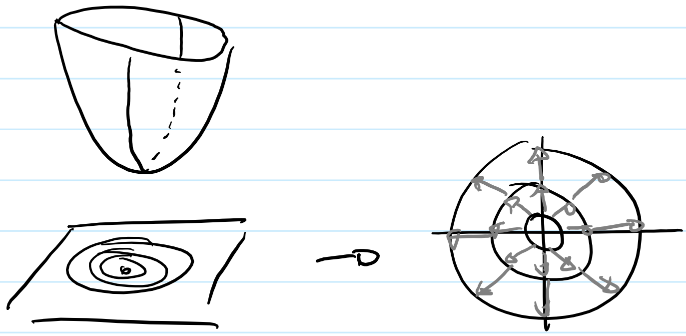


* vectors not to scale

$$\begin{vmatrix} \frac{\partial f}{\partial x} = -2x \\ \frac{\partial f}{\partial y} = -2y \end{vmatrix}$$

$$\vec{\nabla} f = (-2x, -2y)$$
$$= -2(x,y)$$
$$= -2\vec{r}$$

<u>Example</u>
$f(x,y) = 1 + x^2 + y^2$



$\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y$

$\vec{\nabla} f = (2x, 2y)$
$= 2\vec{r}$

Some features of these examples hold in general:

1. The gradient field is everywhere perpendicular to the level curves of f
2. The direction of the gradient is along the path of quickest increase.
3. The magnitude of the gradient is proportional to the rate of quickest increase.

The key to proving 1-3 is the multivariable chain rule:

$g(t) = f(x(t), y(t))$

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}(x(t), y(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\,\frac{dy}{dt}(t)$$

<u>Example</u>

$f(x,y) = x^2 y^3$

for this example

$x(t) = t \qquad \frac{dx}{dt} = 1 \qquad \frac{\partial f}{\partial x} = 2xy^3$

$y(t) = t^2 \qquad \frac{dy}{dt} = 2t \qquad \frac{\partial f}{\partial y} = 3x^2y^2$

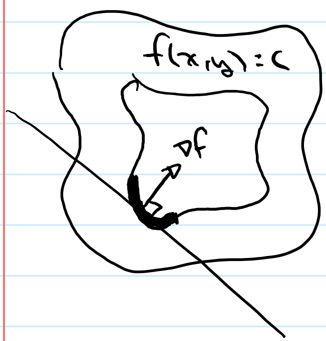
$g(t) = f(t, t^2) = (t)^2(t^2)^3 = t^8$

$\frac{dg}{dt} = 8t^7$

while chain rule says:

$$\begin{aligned} \frac{dg}{dt} &= (2xy^3)(t,t^2)\cdot 1 + (3x^2y^2)(t,t^2)\cdot(2t) \\ &= (2t(t^2)^3)\cdot 1 + (3(t)^2(t^2)^2)(2t) \\ &= 2t^7 + 6t^7 = 8t^7 \end{aligned}$$

Now suppose $t \mapsto (x(t), y(t))$ is a parameterized path that lies on a level curve of f.



$$g(t) = f(x(t), y(t)) = c$$

$$\frac{dg}{dt} = 0 \quad \left(\frac{d}{dt}(c) = 0\right)$$

By chain rule:

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)$$

$$= \vec{\nabla} f(\vec{r}(t)) \cdot \vec{v}(t) \Rightarrow \vec{\nabla} f \text{ is perpendicular to } \vec{v}$$

<u>Conclusion</u>

$\vec{\nabla} f$ is perpendicular to level curves of f.