L6: Vector Fields, Flow Lines

September 22, 2016 1:29 PM

Announcements

- some notes online with more details on the catenary derivation (see course page)
- solutions to HW1 are up (course page)
- Typo in HW2, 3(a)

  $\vec{F}(x,y) = (-4y, x)$, not $\vec{F}(x,y) = (4y, -x)$

Today: Vector Feilds

Q: Imagine that you're trying to describe the flow of a fluid on some region in $\mathbb{R}^2$ (or $\mathbb{R}^3$). What sort of data would describe the flow precisely?

$\to$ 2 real numbers for the coordinates of point
$\to$ 2 real numbers for the components of vector $\Big\} \mathbb{R}^2$

The notion of a vector field is a formalization of this concept.

Def^n:

For a region X in $\mathbb{R}^2$ (or $\mathbb{R}^3$), a vector field is a continuous function $\vec{F}: X \to \mathbb{R}^2$ (or $\mathbb{R}^3$)

In coordinates, this is equivalent to a pair (or group of 3) continuous functions $F_x(x,y,z)$, $F_y(x,y,z)$, $F_z(x,y,z)$

$$\vec{F}(x,y,z) = \big(F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\big)$$

F_x, F_y, F_z are the component functions of the vector field.

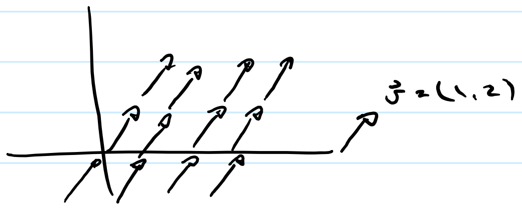
$$(x,y,z) \mapsto \big(F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)\big), \ (x,y,z) \in X$$
denotes the same thing

Examples:

1) Steady flow: Fix a vector $\vec{v} \in \mathbb{R}^2$ (or $\mathbb{R}^3$)

$$\vec{F}(x,y,z) = \vec{v}$$

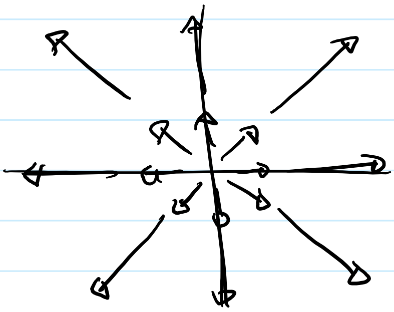
eg.



2) Position vector field

$$\vec{F}(\vec{r}) = \vec{r}$$
$$\vec{F}(x,y,z) = (x,y,z)$$

$$F_x(x,y,z) = x$$
$$F_y(x,y,z) = y$$
$$F_z(x,y,z) = z$$

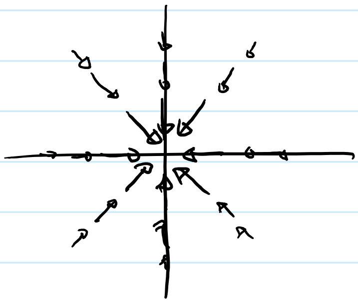


→ zero curl, large divergence

3) Inverse square field

$$\vec{F}(\vec{r}) = -\frac{\vec{r}}{\|\vec{r}\|^3} \Rightarrow \underbrace{\frac{1}{\|\vec{r}\|^2}}_{\text{length of vector or strength of force}} \cdot \underbrace{\frac{\vec{r}}{\|\vec{r}\|}}_{\text{direction vector}}$$

$$F(x,y,z) = \left(-\frac{x}{x^2+y^2+z^2},\ -\frac{y}{x^2+y^2+z^2},\ -\frac{z}{x^2+y^2+z^2}\right)$$

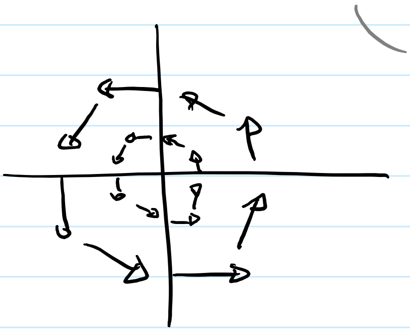


* Gravity and electric attraction due to point masses / charges have this character.

4) Steady rotation in $\mathbb{R}^2$

$$\vec{F}(x,y) = (-y, x)$$

this vector is (x,y) rotated $\pi/2$ rad counterclockwise.



→ large curl, zero divergence

Operations on vector fields

Given two vector fields $\vec{F}$ and $\vec{G}$, and a real number λ, you can define new vector fields:

$$(\vec{F}+\vec{G})(x,y,z) = \vec{F}(x,y,z) + \vec{G}(x,y,z)$$
$$(\lambda\vec{F})(x,y,z) = \lambda\vec{F}(x,y,z)$$

Similarly,

$(\vec{F}\cdot\vec{G})(x,y,z) = \vec{F}(x,y,z)\cdot\vec{G}(x,y,z)$ is a real valued function on X

$(\vec{F}\times\vec{G})(x,y,z) = F(x,y,z) \times \vec{G}(x,y,z)$ is a vector field.

Flow lines

Let $\vec{F}$ be a vector field on $X \subset \mathbb{R}^2$ or $\mathbb{R}^3$

Def^n

A parameterized path $t \mapsto \vec{r}(t)$, $t \in I$ is called a flow line of $\vec{F}$ if:

$$\vec{v}(t) = \vec{F}(\vec{r}(t))$$

In coordinates,

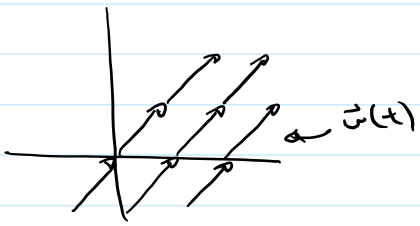
not the derivative

$$\left|\begin{array}{l} x'(t) = F_x(x(t), y(t), z(t)) \\ y'(t) = F_y(x(t), y(t), z(t)) \\ z'(t) = F_z(x(t), y(t), z(t)) \end{array}\right\} \text{system of ODEs}$$

ordinary differential equations

Examples:

1) Steady flow



Guess: Should be straight lines with direction vector $\vec{v}$

$t \mapsto \vec{r}_0 + t\vec{v} = \vec{r}(t)$ is a description of such a line

Let's check this is a flow line:

$$\vec{v}(t) = \vec{v} = \vec{F}(\vec{r}(t))$$

How would you find this more systematically?

$$\vec{v} = (v_x, v_y, v_z) \qquad \vec{F}(x,y,z) = (\underset{F_x}{v_x}, \underset{F_y}{v_y}, \underset{F_z}{v_z})$$

$| x'(t) = v_x \xrightarrow{\text{integrate w/r/t } t} x(t) = v_x t + c$

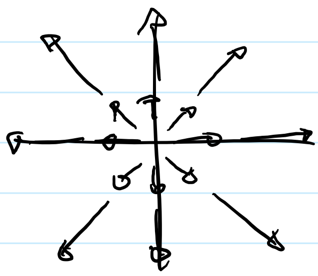
integrate w/r/t t

$$\left|\begin{array}{l} x'(t) = v_x \rightsquigarrow x(t) = v_x t + r_x \\ y'(t) = v_y \rightsquigarrow y(t) = v_y t + r_y \\ z'(t) = v_z \rightsquigarrow z(t) = v_z t + r_z \end{array}\right.$$

write $\vec{r}_0 = (r_x, r_y, r_z)$

$$(x(t), y(t), z(t)) = \vec{r}(t) = \vec{r}_0 + \vec{v}t$$

2) Position field



$$\vec{F}(x,y) = (\overset{F_x}{x}, \overset{F_y}{y})$$

$$\left|\begin{array}{l} x'(t) = x \\ y'(t) = y \end{array}\right. \rightsquigarrow \left|\begin{array}{l} x(t) = Ae^t \\ y(t) = Be^t \end{array}\right.$$

$$\left|\begin{array}{l} x'(t) = Ae^t = x(t) \\ y'(t) = Be^t = y(t) \end{array}\right.$$

If you started at $\vec{r} = (r_x, r_y)$ at $t = 0$

$$\begin{array}{l} Ae^0 = x(0) = r_x \\ Be^0 = y(0) = r_y \end{array} \longrightarrow \begin{array}{l} A = r_x \\ B = r_y \end{array}$$

$$t \mapsto (r_x e^t, r_y e^t), \ t \in \mathbb{R}$$

$$\Downarrow$$

$$e^t(r_x, r_y)$$

* explanation → next lecture