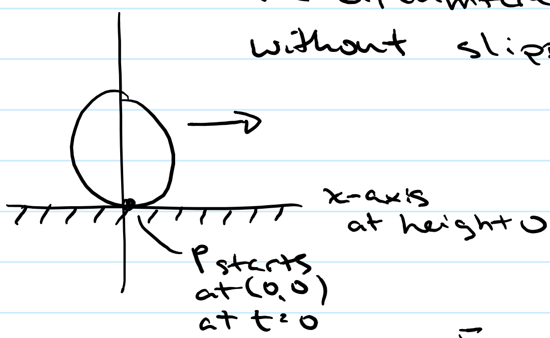
L5: Cycloids, Catenaries

September 21, 2016 11:30 AM

Today: Some examples of curves:
- cycloids
- catenaries.

Cycloids

Problem: Describe the path followed by a point P on the circumference of the unit circle that rolls without slipping along the x-axis.

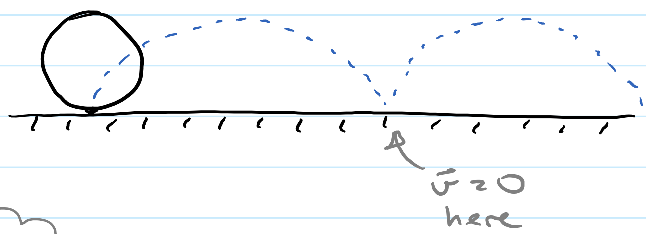


The motion of the centre of the circle is described by:
$t \mapsto (t, 1)$

From the reference frame of the centre of the circle, the point P's motion is described by:
$t \mapsto (-\sin t, -\cos t)$

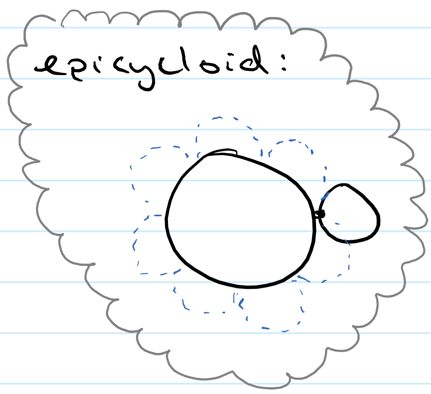
The composite motion is given by adding these two vectors:
$t \mapsto (t - \sin t, 1 - \cos t)$, $t \in [0, \infty)$

motion:



The velocity of P is given by:
$(1 - \cos t, \sin t)$
When $t = 2n\pi$, $n \in \mathbb{Z}$, this is 0!

Zahlen
The integers
$\ldots, -2, -1, 0, 1, 2, \ldots$



What happens to the slope of the path as $t \to 2\pi$?

$$\frac{y'(t)}{x'(t)} = \frac{\sin t}{1 - \cos t} \xrightarrow[t \to 2\pi]{} \frac{0}{0}$$

l'hôpital

$$\lim_{t \to 2\pi} \frac{\cos t}{\sin t} \to \infty$$

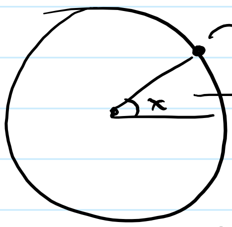
Woah!!

Alternative Parameterization of a branch of the hyperbola:

By analogy with unit circle:

branch of the hyperbola:

By analogy with unit circle:

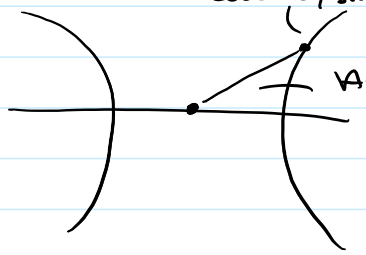


$(\cos x, \sin x)$

Area $= \frac{x}{2}$

$x^2 + y^2 = 1$

$(0 \le x \le 2\pi)$



$(\cosh x, \sinh x)$

Area $= \frac{x}{2}$

$x^2 - y^2 = 1$

$$\left(\begin{array}{c} \cos(x) = \dfrac{e^{ix} + e^{-ix}}{2i} \\ \sin(x) = \dfrac{e^{ix} - e^{-ix}}{2} \\ \text{Where } i = \sqrt{-1} \end{array}\right)$$

$\cos^2 x + \sin^2 x = 1$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \sin x = \cos x$

Solutions to

$y'' = -y$

The functions $\cosh x$ and $\sinh x$ are called the hyperbolic cosine and sine

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$\cosh^2 x - \sinh^2 x = 1$

$\frac{d}{dx} \cosh x = \sinh x$

$\frac{d}{dx} \sinh x = \cosh x$

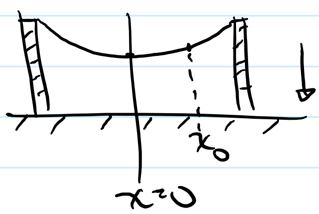
Solutions to

$y'' = y$

Half-angle formulas, angle addition formulas, ...

Catenaries

Problem: Describe the shape of a wire of uniform density ρ that hangs from two supports of equal height under uniform gravity?



gravity

Say the wire is described by the function $f(x)$

(so it's parameterized by $t \mapsto (t, f(t))$)

$\vec{r} = (t, f(t))$

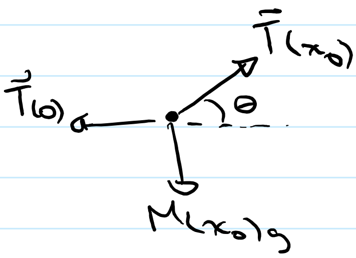
$\vec{v} = (1, f'(t))$

$\|\vec{v}(t)\| = \sqrt{1 + f'(t)^2}$

What are the forces on the peice of wire between $x=0$ and $x = x_0$?

The mass of this piece is equal to:

$$\int_x^{x_0} \rho \sqrt{1 + f'(t)^2}\, dt = M(x_0)$$



What is θ? $T(x_0)$ is tangent to the graph of $f(x)$ at x_0, so

$$\tan\theta = f'(x_0)$$

$$\|\vec{T}(x_0)\|\sin\theta = M(x)g$$
$$\|\vec{T}(x_0)\|\cos\theta = \|\vec{T}(0)\|$$

Divide...

$$f'(x) = \tan\theta = \frac{M(x)g}{\|\vec{T}(0)\|}$$

Differentiate (with respect to) w/r/t x_0 and use Fund. Theo. of Calc. to see:

$$f''(x_0) = \frac{\rho\sqrt{1+f'(x)^2}}{\|\vec{T}(0)\|}$$

$$\frac{\|\vec{T}(0)\|}{\rho g}\cosh\left(\frac{\rho g}{\|\vec{T}(0)\|}x_0\right) + C$$ is a solution to this differential eqn.