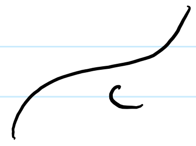
L4: Integrals of Real-Valued Functions Along Paths, Arclength

September 19, 2016 12:26 PM

Integrals of Real-Valued Functions Along Paths

Let C be a curve parameterized by $t \mapsto \vec{r}(t)$, $t \in I$

Let f be a function defined and continuous in a neighborhood of C



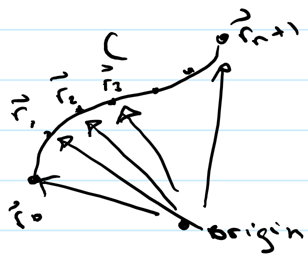
Def

The integral of f along C is

$$\int_{t \in I} f(\vec{r}(t)) \cdot \|\vec{v}(t)\| \, dt$$

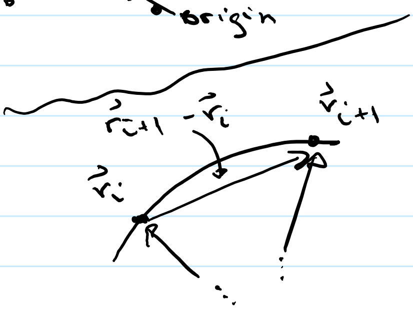
(special case when $f=1$ everywhere on C, get back arc length)

Why this is a reasonable definition:
(say $I = [a,b]$ for concreteness)



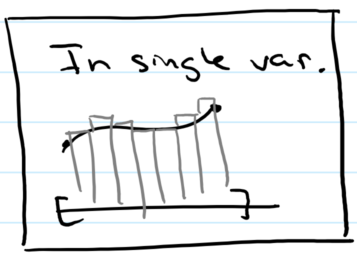
Choose $n+2$ points on C, with position vectors $\vec{r}_i = \vec{r}(t_i)$, with

$$\vec{r}_0 = \vec{r}(a)$$
$$\vec{r}_{n+1} = \vec{r}(b)$$



An estimate for what the integral of f along C should be given by:

$$\sum_{i=1}^{n} f(\vec{r}_i) \|\vec{r}_{i+1} - \vec{r}_i\|$$



Now, $\vec{r}_{i+1} - \vec{r}_i$

$$= \vec{r}(t_{i+1}) - \vec{r}(t_i) \approx \vec{v}(t_i) \cdot (t_{i+1} - t_i)$$

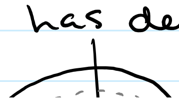
$$\sum_{i=1}^{n} f(\vec{r}(t_i)) \|\vec{v}(t_i)\| (t_{i+1} - t_i)$$

$$\xrightarrow[n \to \infty]{} \int_a^b f(\vec{r}(t)) \|\vec{v}(t)\| \, dt$$

Mass of a bent wire

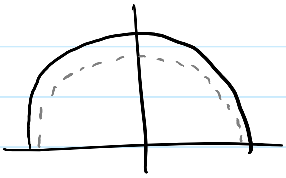
Suppose that a piece of wire having the shape of the upper unit semicircle has density:

$$\delta(x,y) = y$$



upper unit semicircle has density:

$\delta(x,y) = y$



The mass of the wire is given by the integral of δ over the semicircle.

Parameterize C by:

$t \mapsto (\cos t, \sin t), \quad t \in [0, \pi]$

$\vec{v}(t) = (-\sin t, \cos t) \qquad \|\vec{v}(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

$$\int_0^{\pi} \delta(\vec{r}(t))\|\vec{v}(t)\|\,dt = \int_0^{\pi} \delta(\cos t, \sin t)\cdot 1\,dt$$

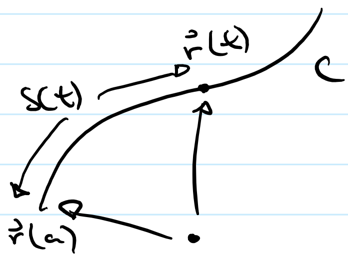
$$= \int_0^{\pi} \sin t\,dt$$

δ only depends on y

Some standard notation for arclength

For a parameterized path $t \mapsto \vec{r}(t), \ t \in I$ define the function

$$s(t) = \int_a^t \|\vec{v}(u)\|\,du$$



By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \|\vec{v}(t)\|$$

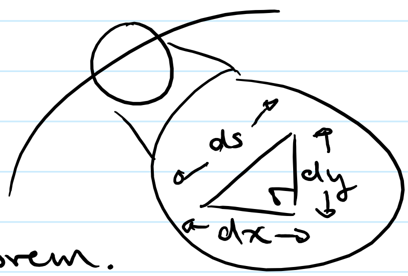
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Classically (and non-rigorously), mathematicians "multiplied by dt" obtaining:

$$ds = \sqrt{dx^2 + dy^2}$$

infinitesimal increase in arclength

infinitesimal increases in x & y



This is reminiscent of Pythagoras' theorem.

Max-min estimate for arclength

Suppose $t \mapsto \vec{r}(t), \ t \in [a,b]$ is a parameterized path

If, $\quad m \le \|\vec{v}(t)\| \le M$, for all t

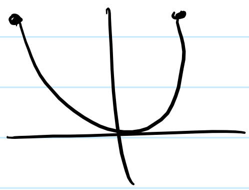
(m, M are real numbers)

Then, $m(b-a) \le \text{arclength}(c) \le M(b-a)$

Proof:

$$\underbrace{\int_a^b m\,dt}_{m\int_a^b dt \,=\, m(b-a)} \le \underbrace{\int_a^b \|\vec{v}(t)\|\,dt}_{\text{arclength}(c)} \le \underbrace{\int_a^b M\,dt}_{M(b-a)}$$

Example



$t \mapsto (t, t^2) \qquad t \in [-1, 1]$

$\vec{v}(t) = (1, 2t)$

$\|\vec{v}(t)\| = \sqrt{1+4t^2}$

Because $t^2 \ge 0$, $1+4t^2 \ge 1$, so $\|\vec{v}(t)\| \ge \underbrace{1}_{=m}$ for all t

Because $t \in [-1, 1]$, $1+4t^2 \le 1+4 = 5$,

$\|\vec{v}(t)\| \le \sqrt{5} = M$

Our estimate gives:

$$1 \cdot (1-(-1)) \le \text{arclength}(\Psi) \le \sqrt{5}\,(1-(-1))$$

$$2 \le \text{arclength}(\Psi) \le 2\sqrt{5}$$

$\text{arclength}(\Psi) \simeq 2.9579$

$2\sqrt{5} \simeq 4.472$

Similarily, if $m \le f(\vec{r}) \le M$ for all points $(\vec{r}) \in C$

then,

$$m \cdot \text{arclength}(C) \le \int_C f\,ds \le M \cdot \text{arclength}(C)$$

the classical notation for the integral of f over C

Next Time:

Shapes of hanging cables