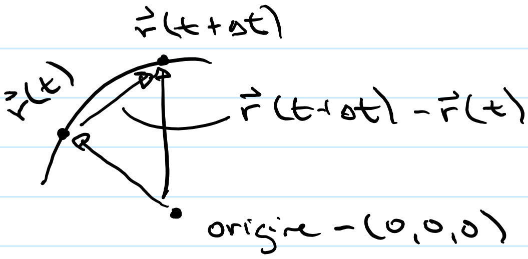
L3: Properties of paths

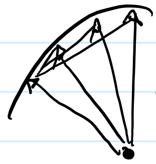
September 15, 2016 1:29 PM

Today: Some properties of paths
- velocity / acceleration
- tangent lines + slopes (in $\mathbb{R}^2$)
- arclength.

For a parameterized path $t \mapsto (x(t), y(t), z(t))$

$\vec{r}(t) = (x(t), y(t), z(t))$ position at time t.
$\vec{v}(t) = (x'(t), y'(t), z'(t))$ velocity
$\vec{a}(t) = (x''(t), y''(t), z''(t))$ acceleration.





$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

Prop

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \vec{v}(t)$$

Proof

$$\lim_{\Delta t \to 0} \frac{(x(t+\Delta t), y(t+\Delta t), z(t+\Delta t)) - (x(t), y(t), z(t))}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t}, \frac{y(t+\Delta t) - y(t)}{\Delta t}, \frac{z(t+\Delta t) - z(t)}{\Delta t} \right)$$

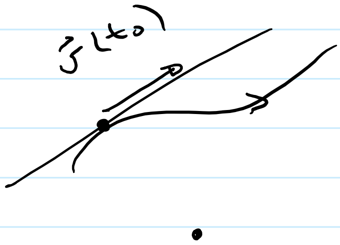
Taking the limit inside, we get:

$$(x'(t), y'(t), z'(t))$$

Principle

For a parameterized path $t \mapsto \vec{r}(t)$ and $t_0 \in I$, $\vec{v}(t_0) \neq 0$, the tangent line to the underlying curve can be parameterized as:

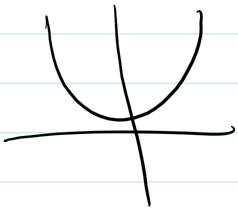
$$u \mapsto \vec{r}(t_0) + u\vec{v}(t_0)$$



Examples

Examples

Parabola:



$t \mapsto (t, t^2) = \vec{r}(t)$

$\vec{v}(t) = (1, 2t)$

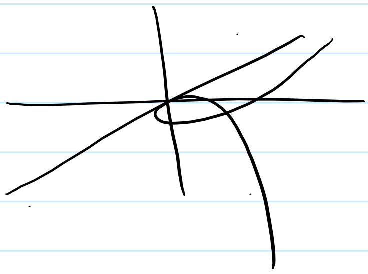
$\vec{r}(0) = (0,0) \qquad \vec{v}(0) = (1,0)$

tangent line: $(0,0) + u(1,0) = (u,0), \quad u \in \mathbb{R}$

$\vec{r}(1) = (1,1) \qquad \vec{v}(1) = (1,2)$

tangent line: $(1,1) + u(1,2) = (1+u, 1+2u), u \in \mathbb{R}$

Twisted Cubic:



$t \mapsto (t, t^2, t^3) = \vec{r}(t)$

$\vec{v}(t) = (1, 2t, 3t^2)$

$\vec{r}(0) = (0,0,0)$

$\vec{v}(0) = (1,0,0)$

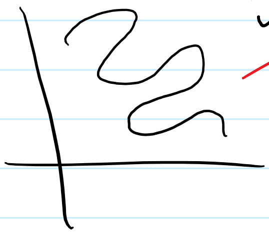
tangent line: x-axis

$\vec{r}(1) = (1,1,1)$

$\vec{v}(1) = (1,2,3)$

$u \mapsto (1+u, 1+2u, 1+3u)$

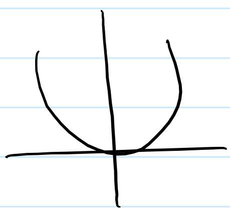
Others:



$\cancel{y = f(x)}$

Def.

For a parameterized path $t \mapsto \vec{r}(t)$ in $\mathbb{R}^2$ define the slope of the underlying curve at time t to be: $\left\{ \frac{y'(t)}{x'(t)}, \; x'(t) \neq 0 \right.$



$y = x^2$

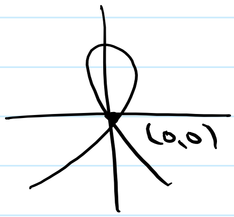
$t \mapsto (t, t^2) = (x(t), y(t))$

$\frac{y'(t)}{x'(t)} = \frac{2 \cdot 1}{1} = 2$

In this case, $x'(t) = 1$ for all t

$y'(t) = f'(x(t))$

$y^2 = x^3 + x^2$



$t \mapsto (t^2-1,\ t(t^2-1))$

The curve passes through the point $(0,0)$ twice at times $t=-1$ and $t=1$

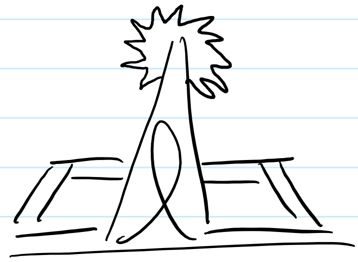
The slopes of the two tangents are:

$x'(t) = 2t$

$y'(t) = 3t^2 - 1$

$$\frac{y'(1)}{x'(1)} = \frac{2}{2} = 1 \qquad \frac{y'(-1)}{x'(-1)} = \frac{3\cdot(-1)^2-1}{2(-1)} = -1$$

Both of the definitions/constructions of tangent line and slope were made for the parameterization. It turns out that both are properties of the underlying curve (invariant under reparameterization).

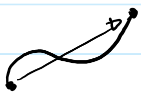


If $t = t(s)$ is a reparameterization

$$\left(\begin{array}{l} \gamma: \bar{I} \to I \\ \qquad s \mapsto t(s) \end{array}\right)$$

$$\begin{array}{l} \frac{d}{ds}\, x(t(s)) = x'(t(s)) \cdot \frac{dt}{ds} \\ \frac{d}{ds}\, y(t(s)) = y'(t(s)) \cdot \frac{dt}{ds} \end{array} \to \vec{v}(s) = \frac{dt}{ds}\,\vec{v}(t)$$

<u>Arclength</u>



<u>Def</u>

For a parameterized path $t \mapsto \vec{r}(t)$, $t \in [a,b]$ arclength of the underlying curve:

$$\int_a^b \|v(t)\|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

(The interval may not be closed or bounded, then make corresponding change in the interval

<u>Ex</u>

Line: $t \mapsto \vec{r}_0 + t\vec{v}$ $\quad t \in [a,b]$

We expect the arc length to equal $\|\vec{v}\|(b-a)$

$\vec{v}(t) = \vec{v}$

$\|\vec{v}(t)\| = \|\vec{v}\|$

$$\int_a^b \|\vec{v}(t)\|\,dt = \int_a^b \|\vec{v}\|\,dt = \|\vec{v}\|\int_a^b dt = \|\vec{v}\|(b-a)$$

Circle: We expect $t \mapsto (R\cos t, R\sin t)$ to have arclength $2\pi R$

$t \in [0, 2\pi]$

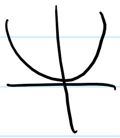
$\vec{r}(t) = (R\cos t, R\sin t)$

$\vec{v}(t) = (-R\sin t, R\cos t)$

$\|\vec{v}(t)\| = \sqrt{(R\sin t)^2 + (R\cos t)^2} = \sqrt{R^2(\sin^2 t + \cos^2 t)} = R$

$\int_0^{2\pi} \|\vec{v}(t)\| dt = \int_0^{2\pi} R dt = R\int_0^{2\pi} dt = 2\pi R$

Parabola:



$t \in [-1, 1]$ $\quad \vec{r}(t) = (t, t^2)$

$t \mapsto (t, t^2)$ $\quad \vec{v}(t) = (1, 2t)$

$\|\vec{v}(t)\| = \sqrt{1 + 4t^2}$

$\int_{-1}^{1} \sqrt{1+4t^2}\, dt \quad \Big| \begin{array}{l} u = 2t \\ du = 2dt \end{array}$

$1 + \tan^2\theta = \sec^2\theta$

$\frac{1}{2}\int_{-2}^{2} \sqrt{1 + u^2}\, du \quad \Big| \begin{array}{l} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{array}$

$\frac{1}{2}\int_{\arctan(-2)}^{\arctan(2)} \sec^3\theta\, d\theta = \Big[\frac{1}{2}(\sec\theta\tan\theta +$

$\ln|\sec\theta + \tan\theta|\Big]_{\arctan(-2)}^{\arctan(2)}$