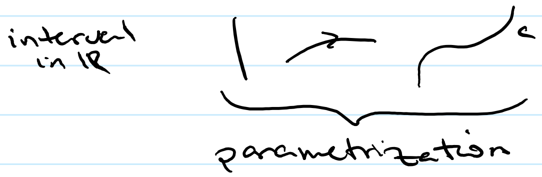
L2: How to write down a function

September 14, 2016 11:28 AM

work $= \int_c \vec{F}\, d\vec{r}$



<u>How to write down function:</u>

Informally, a function is rule that assigns, to every element of a set X, a corresponding element of a set Y

(Reminder: Informally a set is a collection of elements)

$$\begin{aligned} f: X &\to Y \\ x &\mapsto f(x) \end{aligned}$$

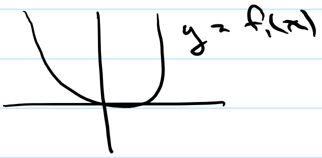
ex. $\underbrace{f(x) = x^5 - 10x}_{\text{ambiguous, need to explicitly define } x}$

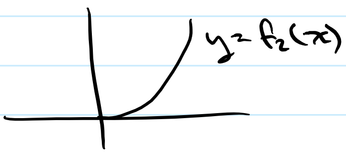
$$\begin{aligned} f_1: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

$(f(x) = x^2)$

$$\begin{aligned} f_2: \mathbb{R}^{\geq 0} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

f_1 and f_2 are different functions, even though they take the same values where they are defined.





X, Y, f can be quite general...

Ex. X = set of all possible strings of form

http:// $\underbrace{\quad\quad}_{\text{word}}$

Y = set of all webpages

$$\begin{aligned} f: X &\to Y \\ \text{URL} &\mapsto \text{webpage} \end{aligned}$$

<u>Parameterized Paths</u>

Reminder: a function from $U \subseteq \mathbb{R}$ to $\mathbb{R}^n$ is given by n functions

$(x_1, \dots, x_n)$

$x_i \in \mathbb{R}$

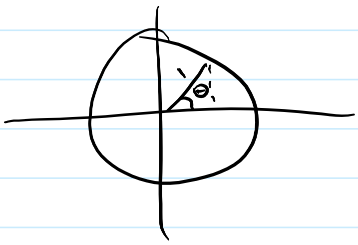
$f_i: U \to \mathbb{R}, \; i = 1, \dots, n$

$$\begin{aligned} f: U &\to \mathbb{R}^n \\ t &\mapsto (f_1(t), \dots, f_n(t)) \end{aligned}$$

f is called differentiable if all of f_i are differentiable.

Example: Suppose we want to describe the unit circle in $\mathbb{R}^2$

$(\cos\theta, \sin\theta)$

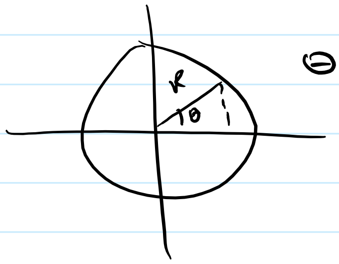


$x^2+y^2=1$

$[0, 2\pi) \to \mathbb{R}^2$

$\theta \longmapsto (\cos\theta, \sin\theta)$

The image of this function is the unit circle.



$\theta \longmapsto (R\cos\theta, R\sin\theta)$

$(R\cos\theta)^2 + (R\sin\theta)^2$

$= R^2(\cos^2\theta + \sin^2\theta)$

$= R^2 \cdot 1 = R^2$

Definition

A parameterized path in $\mathbb{R}^n$ is a differentiable function from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$

Intuitively, this describes the motion of a particle in $\mathbb{R}^n$, with I representing an interval of time.

* For us, $n = 2$ or 3

$x(t) = \ldots$

$y(x) = \ldots \quad , t \in I \quad \Big| \quad t \mapsto (x(t), y(t), z(t))$, $t \in I$

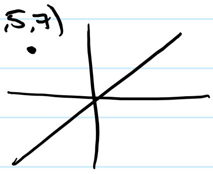
$z(x) = \ldots$

Zoo of examples

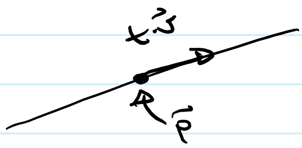
In $\mathbb{R}^2$

- simple possibility: $x(t) = 3$, $y(t) = 5$, $z(t) = 7$ (degree 0)

$(3,5,7)$



- Lines: A line is determined by a point $\vec{p} = (x_0, y_0, z_0)$, and a direction $\vec{v} = (v_1, v_2, v_3)$

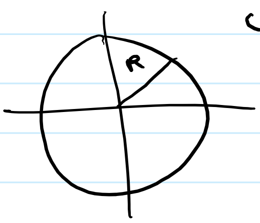


$\vec{p} + t\vec{v}, \; t \in \mathbb{R}$

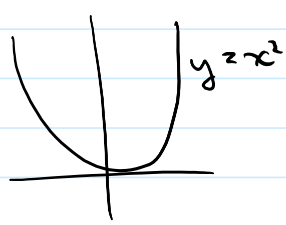
$$\begin{cases} x(t) = x_0 + tv_1 \\ y(t) = y_0 + tv_2 \\ z(t) = z_0 + tv_3 \end{cases}, \; t \in \mathbb{R} \qquad \text{Degree 1}$$

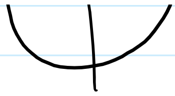
- conics

circle

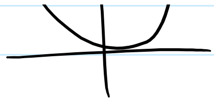


parabola





$t \mapsto (R\cos t, R\sin t)$
$t \in \mathbb{R}$ or $[0, 2\pi)$

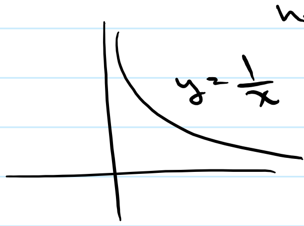


$t \mapsto (t, t^2)$
$t \in \mathbb{R}$

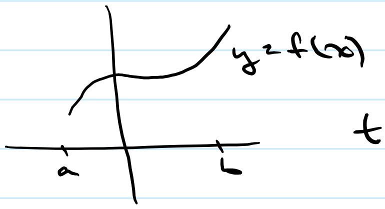
ellipse

$t \mapsto (2\cos t, \sin t)$

hyperbola



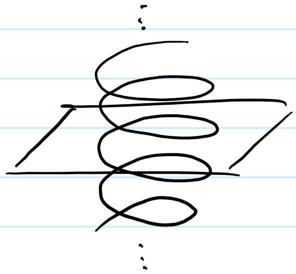
$t \mapsto (t, \frac{1}{t})$
$t \in (0, \infty)$



$t \mapsto (t, f(t))$
$t \in [a, b]$

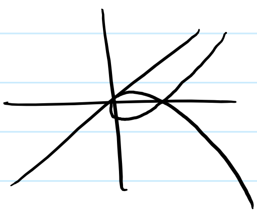
<u>In $\mathbb{R}^3$</u>

$(\cos t, \sin t, t)$
$t \in \mathbb{R}$



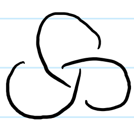
example of a helix

Twisted cubic



$t \mapsto (t, t^2, t^3)$

$$x(t) = (2 + \cos 2t)\sin 3t$$
$$y(t) = (2 + \sin 2t)\sin 3t \quad , \; t \in [0, 2\pi)$$
$$z(t) = \cos 3t$$



<u>Reparametrization</u>

$$x^2 + y^2 = 1$$

Try to solve for y: $y^2 = 1 - x^2$

$= \sqrt{1 - x^2}$

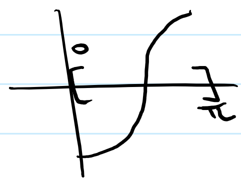
Try to solve for y: $y^2 = 1 - x^2$
$y = \sqrt{1-x^2}$

$t \mapsto (t, \sqrt{1-t^2})$
$t \in [-1, 1]$
$t \mapsto (-\cos t, \sin t)$
$t \in [0, \pi]$

These two parameterized paths have the same <u>underlying curve</u>, the upper semicircle.

<u>Def.</u>
An (orientation-preserving) change of parameter is an increasing, differentiable onto map $\gamma : \tilde{I} \to I$
Given a parameterized path $I \to \mathbb{R}^3$
$t \mapsto (x(t), y(t), z(t))$
get a different path
$\tilde{I} \to \mathbb{R}^3$
$s \mapsto (x(\gamma(s)), y(\gamma(s)), z(\gamma(s)))$

$\tilde{I} = [0, \pi]$
$I = [-1, 1]$
$\gamma : \tilde{I} \to I$
$s \mapsto -\cos(s)$



$I \to \mathbb{R}^2$
$t \mapsto (t, \sqrt{1-t^2})$

Reparameterization:
$\tilde{I} \to \mathbb{R}^2$
$s \mapsto (-\cos(s), \sqrt{1-\cos^2(s)})$
$= (-\cos(s), \sin(s))$

For a parameterized path
$t \mapsto (x(t), y(t), z(t))$
<u>define</u> its velocity at t_0 to be:
$(x'(t_0), y'(t_0), z'(t_0))$
acceleration at t_0 to be:
$(x''(t_0), y''(t_0), z''(t_0))$

<u>Prop.</u>
Given a parameterized path, if $(x'(t_0), y'(t_0), z'(t_0)) \neq 0$, then the tangent line to the underlying curve at t_0

to be parameterized by

$$u \mapsto (x(t_0) + u x'(t_0), y(t_0) + u y'(t_0), z(t_0) + u z'(t_0))$$