MTHE 227 Midterm Examination

October 25 2016, 6:00 pm – 8:00 pm

Instructions. There are six questions on this exam, totaling to 100 points. Attempt each question. The point worth of each question is specified below. The duration of the exam is two hours. Calculators, data sheets and other aids are not permitted. To receive full credit, you must explain your answers (except for the last question). Answers are to be recorded in an exam booklet handed out by the instructor.

Good luck!

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1 (10 points). Let $f(x,y,z) = 15\sqrt{1+4y+9xz}$. Let C be the segment of the twisted cubic curve traced out by $t \mapsto (t, t^2, t^3)$, $t \in [0,1]$. Compute $\int_C f\,ds$.

2. Let C be a unit circle centered at the point $(1,1)$ in $\mathbb{R}^2$.

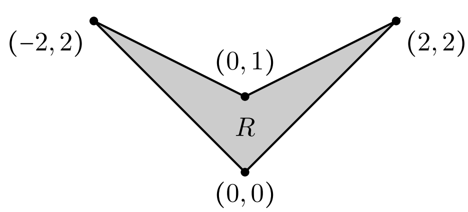
(a) (7 points) Parametrize C.

(b) (8 points) For each point of C, parametrize the line tangent to C at that point.

(If you are having trouble, you can instead parametrize the unit circle centered at the origin, as well as its tangent lines, for a maximum of 10 points.)

3. (a) (6 points) Let R be the rectangle $[0,1]\times[0,1]$ in $\mathbb{R}^2$. Let $f(x,y) = xe^{xy}$. Compute $\iint_R f\,dA$.

(b) (7 points) Let R be the following region in $\mathbb{R}^2$, bounded by four line segments:

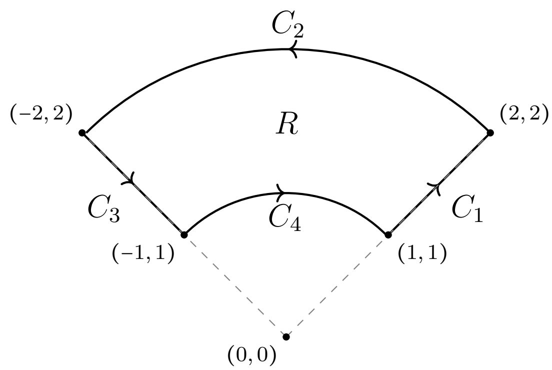


Let $f(x,y) = xy + x + y + 1$. Set up $\iint_R f\,dA$ as an iterated integral, or a sum of iterated integrals. It is not necessary to evaluate the integral.

(c) (7 points) Compute $\displaystyle\int_0^1 \int_{3x}^3 \cos(y^2)\,dydx$. (Suggestion: Change the order of integration. The function $\cos(y^2)$ has no elementary antiderivative.)

4 (15 points). Let C be the line segment connecting the points $(2,0)$ and $(1,6)$ in $\mathbb{R}^2$. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y) = (x,y)$. Compute the flux $\int_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds$ of $\mathbf{F}$ across C, with the normal pointing up and to the right.

5. Let $C = C_1 + C_2 + C_3 + C_4$ be the (oriented and closed) piecewise curve below:



The dashed lines are not part of the curve C. The curves C_1 and C_3 are straight line segments, and the curves C_2 and C_4 are arcs of circles.

Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right).$$

(a) (10 points) Parametrize the curves C_1, C_2, C_3 and C_4, with the orientations indicated by the arrows. For each $i = 1, 2, 3, 4$, compute $\int_{C_i} \mathbf{F} \cdot \mathbf{dr}$. Compute $\int_C \mathbf{F} \cdot \mathbf{dr}$.

(b) (5 points) Does there exist a real-valued function ϕ so that $\mathbf{F} = \nabla \phi$? If so, find such a ϕ; if not, give a reason why not. (Is $\mathbf{F}$ path-independent?)

(c) (5 points) Compute

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}} \right) - \frac{\partial}{\partial y}\left(\frac{-y}{\sqrt{x^2+y^2}} \right).$$

(d) (10 points) Let $f(x,y)$ denote the result of part (c). Let R be the region bounded by C. Compute $\iint_R f\, dA$. (Suggestion: Use polar coordinates.)

Remark. As a check on your answers, you should find that $\int_C \mathbf{F} \cdot \mathbf{dr} = \iint_R f\, dA$ (Green's Theorem). However, you may not apply Green's Theorem in your solution of this question.

6 (10 points)**.** True or False? 2 points each. No justification necessary. No penalty for incorrect answers.

(a) For any vector field $\mathbf{F}$, the value of $\int_C \mathbf{F} \cdot \mathbf{dr}$ depends only on the two endpoints of C.

(b) For any conservative vector field, there exists a unique potential function.

(c) The gradient field of a function is perpendicular to the level curves of that function at every point.

(d) For any parametrization of a curve C, the normal vector $\mathbf{n}_+$ always points away from the origin.

(e) It is possible for a parametrized path to intersect itself at a point different from an endpoint.