MTHE 227 Midterm Solutions

1 (10 points)**.** Let $f(x,y,z) = 15\sqrt{1+4y+9xz}$. Let C be the segment of the twisted cubic curve traced out by $t \mapsto (t, t^2, t^3)$, $t \in [0,1]$. Compute $\int_C f\,ds$.

Solution. The velocity of the parametrization is $\mathbf{v}(t) = (1, 2t, 3t^2)$. The speed is then $\|\mathbf{v}(t)\| = \sqrt{1+4t^2+9t^4}$. We have

$$\begin{aligned}\int_C f\,ds &= \int_0^1 f(x(t),y(t),z(t))\,\|\mathbf{v}(t)\|\,dt \\ &= \int_0^1 15\sqrt{1+4t^2+9t\cdot t^3}\cdot\sqrt{1+4t^2+9t^4}\,dt \\ &= \int_0^1 15(1+4t^2+9t^4)\,dt \\ &= \left[15t+20t^3+27t^5\right]_{t=0}^{t=1} \\ &= 15+20+27 \\ &= 62.\end{aligned}$$

2. Let C be a unit circle centered at the point $(1,1)$ in $\mathbb{R}^2$.

(a) (7 points) Parametrize C.

(b) (8 points) For each point of C, parametrize the line tangent to C at that point.

(If you are having trouble, you can instead parametrize the unit circle centered at the origin, as well as its tangent lines, for a maximum of 10 points.)

Solution.

(a) To parametrize C, we translate every point of the parametrization of the unit circle by adding the vector $(1,1)$. So, one possible parametrization is

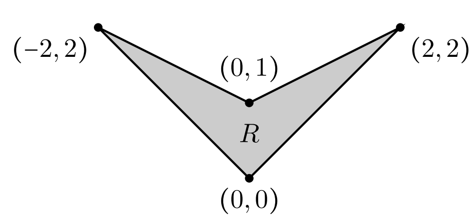
$$(1+\cos(t),\, 1+\sin(t)), \quad t \in [0, 2\pi].$$

(b) The velocity of the parametrization is $\mathbf{v}(t) = (-\sin(t), \cos(t))$. Therefore, the line tangent to C at $(1+\cos(t), 1+\sin(t))$ may be parametrized as

$$u \mapsto (1+\cos(t) - u\sin(t), 1+\sin(t)+u\cos(t)), \quad u \in \mathbb{R}$$

3. (a) (6 points) Let R be the rectangle $[0,1]\times[0,1]$ in $\mathbb{R}^2$. Let $f(x,y) = xe^{xy}$. Compute $\iint_R f\,dA$.

(b) (7 points) Let R be the following region in $\mathbb{R}^2$, bounded by four line segments:



Let $f(x,y) = xy + x + y + 1$. Set up $\iint_R f\,dA$ as an iterated integral, or a sum of iterated integrals. It is not necessary to evaluate the integral.

(c) (7 points) Compute $\int_0^1 \int_{3x}^3 \cos(y^2)\,dydx$. (Suggestion: Change the order of integration. The function $\cos(y^2)$ has no elementary antiderivative.)

Solution.

(a) By Fubini,

$$\begin{aligned}\iint_R f\,dA &= \int_0^1 \left(\int_0^1 xe^{xy}dy\right)dx \\ &= \int_0^1 \left[e^{xy}\right]_{y=0}^{y=1} dx \\ &= \int_0^1 e^x - 1\,dx \\ &= (e-1) - 1 \\ &= e - 2.\end{aligned}$$

(b) We can split the region into two triangles $(0,0)$, $(-2,2)$, $(0,1)$ and $(0,0)$, $(2,2)$, $(0,1)$, both of which are Type I regions (in fact, Type III regions).

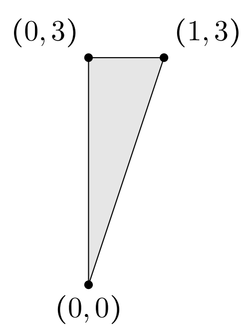
For the left triangle, the top boundary is the line $y = -x/2+1$, and the bottom boundary is the line $y = -x$.

For the right triangle, the top boundary is the line $y = x/2+1$, and the bottom boundary is the line $y = x$.

Therefore, by Fubini, the double integral is

$$\iint_R f(x,y)\,dA = \int_{-2}^0 \int_{-x}^{-x/2+1} xy + x + y + 1\,dydx + \int_0^2 \int_x^{x/2+1} xy + x + y + 1\,dydx.$$

(c) The region of integration is the following triangle:



Therefore, the integral with changed order of integration is

$$\int_0^3 \int_0^{y/3} \cos(y^2)dydx = \int_0^3 \frac{y}{3}\cos(y^2)\,dx = \left[\frac{1}{6}\sin(y^2)\right]_{y=0}^{y=3} = \frac{\sin(9)}{6}.$$

4 (15 points)**.** Let C be the line segment connecting the points $(2,0)$ and $(1,6)$ in $\mathbb{R}^2$. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y) = (x,y)$. Compute the flux $\int_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds$ of $\mathbf{F}$ across C, with the normal pointing up and to the right.

Solution. The line segment can be parametrized by $t \mapsto (2-t, 0+6t)$, $t \in [0,1]$. The velocity of this parametrization is $\mathbf{v}(t) = (-1,6)$. Therefore, $\mathbf{n}_+(t) = (6,1)$ is a normal that points up and to the right.

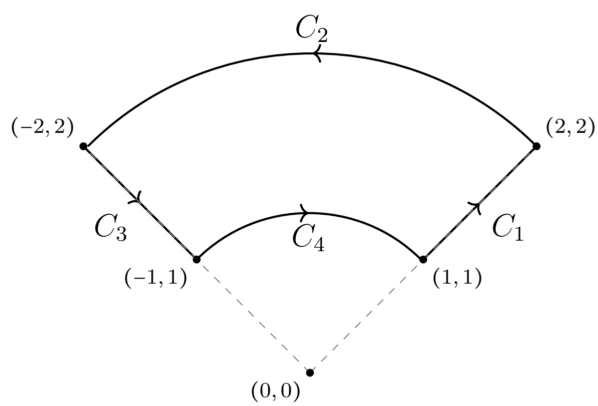
We have

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{n}_+(t) = (2-t,\, 6t)\cdot(6,1) = (2-t)6 + (6t)1 = 12.$$

Therefore,

$$\int_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \int_0^1 12\,dt = 12.$$

5. Let $C = C_1 + C_2 + C_3 + C_4$ be the (oriented and closed) piecewise curve below:



The dashed lines are not part of the curve C. The curves C_1 and C_3 are straight line segments, and the curves C_2 and C_4 are arcs of circles.

Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right).$$

(a) (10 points) Parametrize the curves C_1, C_2, C_3 and C_4, with the orientations indicated by the arrows. For each $i = 1,2,3,4$, compute $\int_{C_i}\mathbf{F}\cdot d\mathbf{r}$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

(b) (5 points) Does there exist a real-valued function ϕ so that $\mathbf{F} = \nabla\phi$? If so, find such a ϕ; if not, give a reason why not. (Is $\mathbf{F}$ path-independent?)

(c) (5 points) Let $f(x,y)$ denote the function

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{\sqrt{x^2+y^2}}\right).$$

Compute $f(x,y)$.

(d) (10 points) Let R be the region bounded by C. Compute $\iint_R f\,dA$. (Suggestion: Use polar coordinates.)

Remark. As a check on your answers, you should find that $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R f\,dA$ (Green's Theorem). However, you may not apply Green's Theorem in your solution of this question.

Solution.

(a) The inner circle has radius $\sqrt{1^2+1^2} = \sqrt{2}$, and the outer circle has radius $\sqrt{2^2+2^2} = \sqrt{8}$. The right angle is $\arctan(\frac{1}{1}) = \arctan(1) = \pi/4$ and the left angle is $\arctan(\frac{-1}{1}) = \arctan(-1) = 3\pi/4$. The four paths may be parametrized as follows:

$$\begin{aligned}
&C_1 : t \mapsto (1+t, 1+t), \quad t \in [0,1],\\
&C_2 : t \mapsto (\sqrt{8}\cos t, \sqrt{8}\sin t), \quad t \in [\pi/4,\, 3\pi/4],\\
&C_3 : t \mapsto (2-t, 2-t), \quad t \in [0,1]\\
&C_4 : t \mapsto (-\sqrt{2}\cos t, \sqrt{2}\sin t), \quad t \in [\pi/4, 3\pi/4].
\end{aligned}$$

The velocity vectors are

$$\begin{aligned}
&C_1 : (1,1),\\
&C_2 : (-\sqrt{8}\sin t, \sqrt{8}\cos t),\\
&C_3 : (-1,-1),\\
&C_4 : (\sqrt{2}\sin t, \sqrt{2}\cos t).
\end{aligned}$$

Therefore, the dot products $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}(t)$ are

$$\begin{aligned}
&C_1 : \left(\frac{-(1+t)}{\sqrt{(1+t)^2+(1+t)^2}}, \frac{1+t}{\sqrt{(1+t)^2+(1+t)^2}}\right)\cdot(1,1) = 0,\\
&C_2 : \left(\frac{-\sqrt{8}\sin t}{\sqrt{8}}, \frac{\sqrt{8}\cos t}{\sqrt{8}}\right)\cdot(-\sqrt{8}\sin t, \sqrt{8}\cos t) = \sqrt{8}(\sin^2 t + \cos^2 t) = \sqrt{8},\\
&C_3 : \left(\frac{-(2-t)}{\sqrt{(2-t)^2+(2-t)^2}}, \frac{2-t}{\sqrt{(2-t)^2+(2-t)^2}}\right)\cdot(-1,-1) = 0,\\
&C_4 : \left(-\frac{\sqrt{2}\sin t}{\sqrt{2}}, -\frac{\sqrt{2}\cos t}{\sqrt{2}}\right)\cdot(\sqrt{2}\sin t, \sqrt{2}\cos t) = -\sqrt{2}(\sin^2 t + \cos^2 t) = -\sqrt{2}.
\end{aligned}$$

So that,

$$\begin{aligned}
&\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 0\,dt = 0,\\
&\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\pi/4}^{3\pi/4} \sqrt{8}\,dt = \sqrt{8}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \sqrt{8}\,\frac{\pi}{2},\\
&\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 0\,dt = 0,\\
&\int_{C_4} \mathbf{F}\cdot d\mathbf{r} = \int_{\pi/4}^{3\pi/4} -\sqrt{2}\,dt = -\sqrt{2}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = -\sqrt{2}\,\frac{\pi}{2}.
\end{aligned}$$

Finally,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \left(\sqrt{8}-\sqrt{2}\right)\frac{\pi}{2}.$$

Remark. This computation could be done in polar coordinates.

The vector field is

$$\mathbf{F}(x,y) = \frac{-y}{\sqrt{x^2+y^2}}\,\mathbf{e}_x + \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{e}_y.$$

This vector field is tangent to circles counterclockwise and has magnitude 1 everywhere. Therefore, in polar coordinates,

$$\mathbf{F}(r,\theta) = \mathbf{e}_\theta(r,\theta).$$

We could verify this using the conversion formulas

$$\begin{aligned}\mathbf{e}_x(r,\theta) &= \cos\theta\,\mathbf{e}_r(r,\theta) - \sin\theta\,\mathbf{e}_\theta(r,\theta)\\ \mathbf{e}_y(r,\theta) &= \sin\theta\,\mathbf{e}_r(r,\theta) + \cos\theta\,\mathbf{e}_\theta(r,\theta).\end{aligned}$$

We have

$$\begin{aligned}\mathbf{F}(r,\theta) = &\frac{-r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}(\cos\theta\,\mathbf{e}_r(r,\theta) - \sin\theta\,\mathbf{e}_\theta(r,\theta)) + \cdots\\ &\cdots + \frac{r\cos\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}(\sin\theta\,\mathbf{e}_r(r,\theta) + \cos\theta\,\mathbf{e}_\theta(r,\theta))\\ &= \mathbf{e}_\theta(r,\theta).\end{aligned}$$

The four paths can be parametrized as

$$\begin{aligned}&C_1 : t \mapsto (t,\,\pi/4),\ \ t\in[\sqrt{2},\,\sqrt{8}],\\ &C_2 : t \mapsto (\sqrt{8},\,\pi/4+t),\ \ t\in[0,\pi/2],\\ &C_3 : t \mapsto (\sqrt{8}-t,\,3\pi/4),\ \ t\in[0,\sqrt{2}],\\ &C_4 : t \mapsto (\sqrt{2},\ 3\pi/4-t),\ \ t\in[0,\pi/2].\end{aligned}$$

The expression for the velocity in polar coordinates is $\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(r,\,\theta) + r\,\frac{d\theta}{dt}\,\mathbf{e}_\theta(r,\theta)$. Therefore,

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{v}(t) = F_r(r(t),\theta(t))\frac{dr}{dt} + F_\theta(r(t),\,\theta(t))r\frac{d\theta}{dt}$$

(note that F_r and F_θ are the r and θ components of $\mathbf{F}$, respectively, not partial derivatives) and so the work is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b F_r(r,\,\theta)\,\frac{dr}{dt} + F_\theta(r,\,\theta)\,r\frac{d\theta}{dt}\,dt.$$

We have,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{\sqrt{2}}^{\sqrt{8}} 0\cdot 1 + 1\cdot t\cdot 0\, dt = 0,$$
$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{0}^{\pi/2} 0\cdot 0 + 1\cdot \sqrt{8}\cdot 1\, dt = \sqrt{8}\cdot \pi/2,$$
$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_{0}^{\sqrt{2}} 0\cdot (-1) + 1\cdot (\sqrt{8}-t)\cdot 0\, dt = 0,$$
$$\int_{C_4} \mathbf{F}\cdot d\mathbf{r} = \int_{0}^{\pi/2} 0\cdot 0 + 1\cdot \sqrt{2}\cdot (-1)\, dt = -\sqrt{2}\cdot \pi/2.$$

So that

$$\int_{C} \mathbf{F}\cdot d\mathbf{r} = \left(\sqrt{8}-\sqrt{2}\right)\frac{\pi}{2},$$

as before.

(b) Here are two possible paths starting and ending at $(2,2)$. One is simply the constant path that stays at the point $(2,2)$; there is no work done along such a path. The other is C, and we found that the work done by $\mathbf{F}$ around C is nonzero. Therefore, $\mathbf{F}$ is not path-independent and cannot be conservative.

Stated another way, by the fundamental theorem of calculus for line integrals, for paths C starting and ending at P, we must have

$$\int_{C} \nabla f\cdot d\mathbf{r} = f(P) - f(P) = 0,$$

but this is not satisfied by C.

For yet another argument, one could compare the work done along different parts of C. For instance, take $-C_1$ and $C_2 + C_3 + C_4$. Both are paths from $(2,2)$ to $(1,1)$, but there is no work done along the first and work done along the second.

(c) For the partial with respect to x, we have

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) &= \frac{\sqrt{x^2+y^2} - x(1/2)(x^2+y^2)^{-1/2}(2x)}{x^2+y^2}\\ &= \frac{(x^2+y^2)-x^2}{(x^2+y^2)^{3/2}}\\ &= \frac{y^2}{(x^2+y^2)^{3/2}}.\end{aligned}$$

The partial with respect to y is found similarly to be $-x^2/(x^2+y^2)^{3/2}$. Therefore,

$$f(x,y) = \frac{x^2+y^2}{(x^2+y^2)^{3/2}} = (x^2+y^2)^{-1/2}.$$

(d) The integral in polar coordinates is

$$\begin{aligned}\int_{\pi/4}^{3\pi/4}\int_{\sqrt{2}}^{\sqrt{8}} \frac{1}{r}\, r\, dr d\theta &= \int_{\pi/4}^{3\pi/4}\int_{\sqrt{2}}^{\sqrt{8}} dr d\theta \\ &= \int_{\pi/4}^{3\pi/4} (\sqrt{8}-\sqrt{2})\, d\theta \\ &= (\sqrt{8}-\sqrt{2})\,[\theta]_{\theta=\pi/4}^{\theta=3\pi/4} \\ &= (\sqrt{8}-\sqrt{2})\frac{\pi}{2}.\end{aligned}$$

6 (10 points)**.** True or False? 2 points each. No justification necessary. No penalty for incorrect answers.

(a) For any vector field $\mathbf{F}$, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the two endpoints of C.

(b) For any conservative vector field, there exists a unique potential function.

(c) The gradient field of a function is perpendicular to the level curves of that function at every point.

(d) For any parametrization of a curve C, the normal vector $\mathbf{n}_+$ always points away from the origin.

(e) It is possible for a parametrized path to intersect itself.

Solution.

(a) False. This is true only for conservative fields. There are many counterexamples; the one we have seen in class is the field $\mathbf{F}(x,y) = (y,0)$, and paths C_1 — line segment from $(-1,0)$ to $(1,0)$ along the x-axis — and C_2 — the top unit semicircle from from $(-1,0)$ to $(1,0)$.

(b) False. Adding any constant to a potential function gives another potential function for the same vector field, so potential functions are not unique.

(c) True. We have proved this in class.

(d) False. Consider, for instance, a clockwise parametrization of a circle.

(e) True. For instance, $t \mapsto (t(t^2-1),\, t^2-1)$ passes through $(0,0)$ at $t=-1$ and $t=1$.