MTHE 227 Practice Midterm Solutions

1 (10 points)**.** Let $f(x,y,z) = 3\sqrt{1+4x^2+4z}$. Let C be the segment of the space parabola traced out by $t \mapsto (t,\, t^2,\, t^2)$, $t \in [0,1]$. Compute $\int_C f\, ds$.

Solution. The velocity of the given path is $\mathbf{v}(t) = (1,\, 2t,\, 2t)$, hence its speed is $\|\mathbf{v}(t)\| = \sqrt{1+(2t)^2+(2t)^2} = \sqrt{1+8t^2}$. Then, we have

$$\begin{aligned}
\int_C f\, ds &= \int_0^1 f(x(t),\, y(t),\, z(t))\, \sqrt{1+8t^2}\, dt \\
&= \int_0^1 3\sqrt{1+4(t)^2+4(t^2)}\sqrt{1+8t^2}\, dt \\
&= \int_0^1 3(1+8t^2)\, dt \\
&= \left[3t + 8t^2\right]_{t=0}^{t=1} \\
&= 3+8 \\
&= 11.
\end{aligned}$$

2. Let C be the ellipse $(x^2/4) + (y^2/9) = 1$ in $\mathbb{R}^2$.

(a) (7 points) Parametrize C.

(b) (8 points) For each point of C, parametrize the line tangent to C at that point.

(If you are having trouble, you can instead parametrize the circle $x^2+y^2 = R^2$, as well as its tangent lines, for a maximum of 10 points.)

Solution.

(a) One possible parametrization is $t \mapsto (2\cos t,\, 3\sin t)$, $t \in [0, 2\pi]$.

(b) At a fixed t_0, the velocity of the above parametrization is $\mathbf{v}(t_0) = (-2\sin t_0,\, 3\cos t_0)$. Therefore, a tangent line to C at $\mathbf{r}(t_0)$ may be parametrized by

$$u \mapsto \mathbf{r}(t_0) + u\mathbf{v}(t_0),\ u \in \mathbb{R}.$$

Written out more explicitly, this is $u \mapsto (2\cos t_0 - 2u\sin t_0,\ 3\cos t_0 + 3u\sin t_0)$, $u \in \mathbb{R}$.

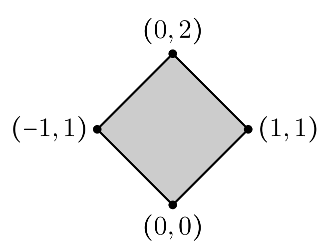
3 (15 points)**.** Let C be the segment of the hyperbola $xy = 4$ from $(1,4)$ to $(4,1)$ in $\mathbb{R}^2$. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y) = (x+y,\ x-y)$. Compute the work $\int_C \mathbf{F}\cdot d\mathbf{r}$ done by $\mathbf{F}$ along C.

Solution. The segment of the hyperbola is the graph of the function $y = 4/x$ over the interval $[1,4]$. Therefore, it can be parametrized as $t \mapsto (t,\ 4/t) =: \mathbf{r}(t)$. The velocity of the parametrization is $\mathbf{v}(t) = (1,\ -4/t^2)$. The integral is then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_1^4 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{v}(t)\,dt \\ &= \int_1^4 \left(t+\frac{4}{t},\ t-\frac{4}{t}\right)\cdot(1,\ -\frac{4}{t^2})\,dt \\ &= \int_1^4 \left(t+\frac{4}{t}-\frac{4}{t}+\frac{16}{t^3}\right)dt \\ &= \left[\frac{t^2}{2}-\frac{8}{t^2}\right]_{t=1}^{t=4} \\ &= \left(\frac{16}{2}-\frac{8}{16}\right)-\left(\frac{1}{2}-\frac{8}{1}\right) \\ &= 8-1/2-1/2+8 \\ &= 15.\end{aligned}$$

4. (a) (6 points) Let R be the rectangle $[0,1]\times[0,1]$ in $\mathbb{R}^2$. Let $f(x,y) = y\cos(\pi xy)$. Compute $\iint_R f\,dA$.

(b) (7 points) Let R be the following quadrilateral in $\mathbb{R}^2$:



Let $f(x,y) = xy+x+y+1$. Set up $\iint_R f\,dA$ as an iterated integral, or a sum of iterated integrals. It is not necessary to evaluate the integral.

(c) (7 points) Compute $\displaystyle\int_0^2 \int_{y/2}^1 e^{-x^2}\,dxdy$. (Suggestion: Change the order of integration. The function e^{-x^2} has no elementary antiderivative.)

Solution.

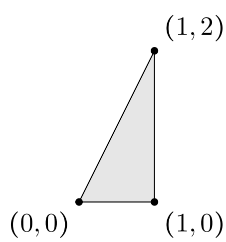
(a) By Fubini's theorem for rectangles, the integral may be computed as (it is better to do the integral with respect to x first, because of the y term outside of the cosine):

$$\begin{aligned}\int_0^1 \int_0^1 y\cos(\pi xy)\,dxdy &= \int_0^1 \left[\frac{1}{\pi}\sin(\pi xy)\right]_{x=0}^{x=1} dy \\ &= \frac{1}{\pi}\int_0^1 \sin(\pi y) - \sin(0)\,dy \\ &= \frac{1}{\pi^2}\left[-\cos(\pi y)\right]_{y=0}^{y=1} \\ &= \frac{1}{\pi^2}\left(-\cos(\pi) + \cos(0)\right) \\ &= \frac{2}{\pi^2}.\end{aligned}$$

(b) We break the region into two triangles with vertices $(-1,1)$, $(0,2)$, $(0,0)$ and $(1,1)$, $(0,2)$, $(0,0)$ and use Fubini's theorem. The boundaries of the left triangle are segments of lines $y = x+2$ (top boundary) and $y = -x$ (bottom boundary). The boundaries of the right triangle are segments of lines $y = -x+2$ (top boundary) and $y = x$ (bottom boundary). Therefore, Fubini's theorem tells us that

$$\iint_R f\,dA = \int_{-1}^0 \int_{-x}^{x+2} f(x,y)\,dydx + \int_0^1 \int_x^{-x+2} f(x,y)\,dydx.$$

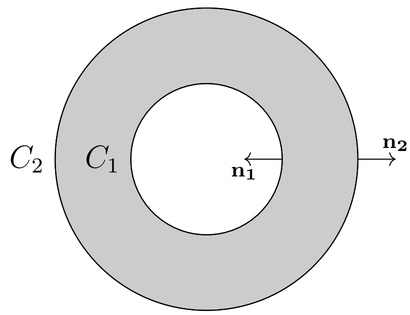
(c) The integral is impossible as written, so we must change the order of integration. The region of integration is a triangle:



Switching the order of integration, we get

$$\begin{aligned}\int_0^1 \int_0^{2x} e^{-x^2}\,dydx &= \int_0^1 2xe^{-x^2}\,dx \\ &= \left[-e^{-x^2}\right]_{x=0}^{x=1} \\ &= 1 - e^{-1}\end{aligned}$$

5. Let C_1 be the circle $x^2 + y^2 = 1$, with normal pointing toward the origin, and C_2 the circle $x^2 + y^2 = 4$, with normal pointing away from the origin.



Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \left(xe^{x^2+y^2},\ ye^{x^2+y^2} \right).$$

(a) (15 points) Parametrize the curves C_1 and C_2. Compute $\int_{C_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds + \int_{C_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$.

(b) (5 points) Let $f(x,y)$ denote the function

$$\frac{\partial}{\partial x}\left(xe^{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(ye^{x^2+y^2}\right).$$

Compute $f(x,y)$.

(c) (10 points) Let R be the region bounded by C_1 and C_2. Compute $\iint_R f\, dA$. (Suggestion: Use polar coordinates. You may need to integrate one of the terms by parts.)

Remark. As a check on your answers, you should find that $\int_{C_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds + \int_{C_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \iint_R f\, dA$ (Green's Theorem). However, you may not apply Green's Theorem in your solution of this question.

(a) The inner circle C_1 may be parametrized by $t \mapsto (\cos t, \sin t)$, $t \in [0, 2\pi]$. The velocity is $\mathbf{v}(t) = (-\sin t, \cos t)$ and $\mathbf{n}_-(t) = (-\cos t, -\sin t)$ is the required normal. We then have

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds &= \int_0^{2\pi} (\cos t\, e,\ \sin te) \cdot (-\cos t,\ -\sin t)\, dt \\ &= \int_0^{2\pi} -e(\cos^2 t + \sin^2 t)\, dt \\ &= -2\pi\, e.\end{aligned}$$

The outer circle C_2 may be parametrized by $t \mapsto (2\cos t, 2\sin t)$, $t \in [0, 2\pi]$. The velocity is $\mathbf{v}(t) = (-2\sin t, 2\cos t)$ and $\mathbf{n}_+(t) = (2\cos t, 2\sin t)$ is the required normal.

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds &= \int_0^{2\pi} \left(2\cos t\, e^4,\ 2\sin te^4\right) \cdot (2\cos t,\ 2\sin t)\, dt \\ &= \int_0^{2\pi} 4e^4(\cos^2 t + \sin^2 t)\, dt \\ &= 2\pi\, 4e^4.\end{aligned}$$

Therefore,

$$\int_{C_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,ds + \int_{C_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = 2\pi\left(4e^4 - e\right).$$

(b) We have

$$\begin{aligned}\frac{\partial}{\partial x}\left(xe^{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(ye^{x^2+y^2}\right) &= \left(e^{x^2+y^2} + xe^{x^2+y^2}(2x)\right) + \left(e^{x^2+y^2} + ye^{x^2+y^2}(2y)\right)\\ &= 2e^{x^2+y^2} + 2(x^2+y^2)e^{x^2+y^2}\end{aligned}$$

(c) In polar coordinates,

$$f(r,\theta) = 2e^{r^2} + 2r^2e^{r^2}$$

and the region of integration is described by $1 \le r \le 2,\ 0 \le \theta \le 2\pi$. Therefore,

$$\iint_R f\,dA = \int_0^{2\pi}\int_1^2 \left(2e^{r^2} + 2r^2e^{r^2}\right) r\,dr\,d\theta.$$

To find the second integrand, use parts:

$$\int r^2(2re^{r^2})\,dr = r^2e^{r^2} - \int 2re^{r^2}\,dr$$

and notice that the second term on the right cancels out the first term in $\iint_R f\,dA$. Therefore,

$$\begin{aligned}\int_0^{2\pi}\int_1^2 2re^{r^2} + 2r^3e^{r^2}\,dr\,d\theta &= \int_0^{2\pi}\left[r^2e^{r^2}\right]_{r=1}^{r=2} d\theta\\ &= 2\pi(4e^4 - e),\end{aligned}$$

agreeing with the result of part (a)!

6 (10 points)**.** True or False? 2 points each. No justification necessary. No penalty for incorrect answers.

(a) The polar coordinates of every point of $\mathbb{R}^2$ are unique (for every point of $\mathbb{R}^2$, there is a unique pair (r,θ) such that (r,θ) are the polar coordinates of that point).

(b) For any conservative vector field, there exists a unique potential function.

(c) The value of the work done by a force represents the change in kinetic energy caused by that force between the two endpoints of the path.

(d) The flux of a vector field that is everywhere parallel to a path across that path is equal to zero.

(e) Any parametrized path has exactly two possible orientations.

Solution.

(a) This is false. For instance, all of the following pairs (r, θ) describe the same point in the plane: $(1,0) = (1,2\pi) = (1,4\pi) = (1,6\pi) = \cdots$.

(b) This is false. If $\mathbf{F} = \nabla f$, then $\mathbf{F} = \nabla(f + C)$ for any real number C, since $\nabla f = \nabla(f + C)$.

(c) This is true. It is an informal statement of the Work-Energy theorem, discussed in the very first lecture.

(d) This is true. If the vector field is everywhere parallel to the path, meaning parallel to the tangent vectors of the path, the vector field is perpendicular to the normal vectors of the path, hence has no flux across the curve.

(e) This is false in general. The 'figure 8' curve provides a counterexample, as discussed in lecture. It is true for simple paths, however, which may be a source of confusion.