MTHE 227 Problem Set 11

Due Thursday December 01 2016 at the beginning of class

1. As a reminder, a torus with radii a and b is the surface of revolution of the circle $(x-b)^2+z^2=a^2$ in the xz-plane about the z-axis (a and b are positive real numbers, with $b>a$). (For two pictures of a torus, see the last page of this problem set.)

(a) Find a function $f(r,\theta,z)$ and a constant $c\in\mathbb{R}$ so that the equation $f(r,\theta,z)=c$ in cylindrical coordinates describes the torus with radii a and b.

(b) Set up two triple integrals in cylindrical coordinates for the volume of the solid torus (the three-dimensional region bounded by a torus) with radii a and b : one with order of integration $dr\,dz\,d\theta$ and the other with order of integration $dz\,dr\,d\theta$.

(c) Check that the volume of the solid torus is equal to $(\pi a^2)(2\pi b)=2\pi^2a^2b$. (It is only necessary to integrate using one of the orders of part (b).)

(You may need to make a sin / cos-type trigonometric substitution.)

2. Find the volume of the region bounded by the surface $z=x^2/4$, and the three planes $y=0$, $y=\ell$ and $z=H$ in $\mathbb{R}^3$, as a function of ℓ and H.

3. Imagine a pool of still fluid (in other words, the fluid is static and in equilibrium). Let h denote the vertical coordinate, measured down from the surface of the fluid, and let x and y denote the usual Cartesian coordinates. As you likely know, if the fluid is incompressible (this is true of water, to a good approximation), the pressure exerted by the fluid varies as[1]

$$p(h,y,z)=\delta gh,$$

where δ is the density of the fluid (assumed uniform), and g is the gravitational constant.

Because of the pressure difference at different heights, a region submerged in the fluid will have a net upward force on it, called the buoyant force, which may be computed as follows.

Let S be a closed (smooth, orientable) surface submerged in the fluid, bounding a region R, and choose inward pointing normals. A small piece of S around the point (x,y,h) with area ΔA will have a force directed perpendicular to it and equal in magnitude (to a good approximation) to $p(x,y,h)\,\Delta A$ (this is just the definition of pressure). To find its component directed up, we can compute the dot product

$$-\mathbf{e_h}\cdot\big((p(x,y,h)\Delta A)\ \hat{\mathbf{N}}(x,y,h)\big)=(-\delta gh\,\mathbf{e_h})\cdot\hat{\mathbf{N}}(x,y,h)\Delta A$$

(the negative sign before $\mathbf{e}_h$ is necessary because of the convention that h points down).

Defining the vector field

$$\mathbf{B}(x,y,h):=(0,0,-\delta gh)=-\delta gh\,\mathbf{e}_h,$$

[1] Instructor's note: On the other hand, if you do not know why, and are curious why, ask me!

and taking $\Delta A \to 0$, the buoyant force on S is therefore equal to the integral

$$\text{Buoyant Force} = \iint_S \mathbf{B} \cdot \hat{\mathbf{N}}\, dS = \iint_S \mathbf{B} \cdot \mathbf{dS}.$$

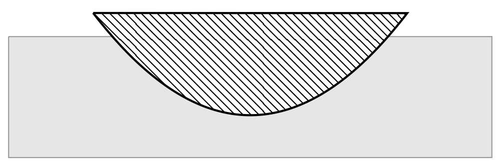
(a) Prove the following theorem, applying the divergence theorem:

Theorem (Archimedes)**.** *The buoyant force on S is equal to the weight of the fluid displaced by S.*

(Take care with the orientation of $\hat{\mathbf{N}}$. In the statement, weight is the product of mass and the gravitational constant g.)

(b) Justify using (b): If R is a region of uniform density d placed in the pool, it will rise if $d < \delta$ and sink if $d > \delta$.

Optional Problem. Let R be a ship modeled as a solid of the kind looked at in Problem 2, of mass $1,080,000\,\text{kg}$, length $\ell = 30\,\text{m}$ and height $H = 10\,\text{m}$. Take the fluid to be water (so, with density $\delta = 1,000\,\text{kg/m}^3$). When the ship is floating at the surface of the water, where will the water level be (measured from the bottom of the ship)?



4. Let R be the region $1 \le x^2 + y^2 \le 9,\ 0 \le z \le 2$ in $\mathbb{R}^3$, and let S be its boundary surface, oriented outward from R. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (2x,\ xy^2,\ xyz).$$

(a) Sketch R. Notice that the boundary surface S splits into four pieces.

(b) Compute the flux integral $\iint_S \mathbf{F} \cdot \mathbf{dS}$ directly, by parametrizing each of the four pieces and computing the flux of $\mathbf{F}$ across each.

(c) Compute $\operatorname{div} \mathbf{F}$, and compute the triple integral $\iiint_R \operatorname{div} \mathbf{F}\, dV$ directly. The answer should be equal to that of part (b) by the divergence theorem.

5. As a reminder, spherical coordinates on $\mathbb{R}^3$ are given by the following map $D \to \mathbb{R}^3_{(x,y,z)}$:

$$\begin{aligned} x(\rho, \theta, \phi) &= \rho \cos(\theta) \sin(\phi), \\ y(\rho, \theta, \phi) &= \rho \sin(\theta) \sin(\phi), \\ z(\rho, \theta, \phi) &= \rho \cos(\phi), \end{aligned}$$

where

$$D = \{(\rho, \theta, \phi) \in \mathbb{R}^3 : \rho \ge 0,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}.$$

Check that

$$\left|\det \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| := \left|\det \begin{pmatrix} \partial x/\partial\rho & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial\rho & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial\rho & \partial z/\partial\theta & \partial z/\partial\phi \end{pmatrix}\right| = \rho^2 \sin(\phi),$$

where $|\cdot|$ denotes the absolute value.

Just for fun (No need to hand-in). Back to the torus! We have seen that, parametrizing the generating circle of the torus with radii a and b by

$$t \mapsto (b + a\cos(t),\ a\sin(t)),\ t \in [0, 2\pi],$$

the torus may be parametrized by

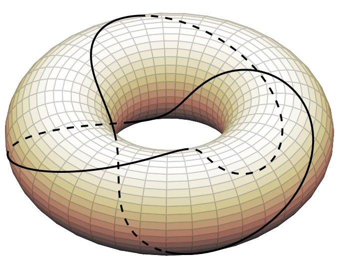
$$\sigma : (\theta, t) \mapsto ((b + a\cos(t))\cos(\theta),\ (b + a\cos(t))\sin(\theta),\ a\sin(t)), \quad \theta \in [0, 2\pi],\ t \in [0, 2\pi].$$

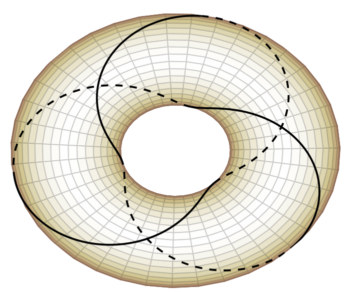
(This is likely a special case of the parametrization of the surface of revolution of a general parametrized curve that you found in Problem Set 9.)

Allow θ and t in the parametrization of a torus to be arbitrary real numbers, disregarding the requirement that a parametrization of a surface be one-to-one in its interior.

Let $a = 1$ and $b = 2$. Write out the path $s \mapsto \sigma(2s,\ 3s),\ s \in [0, 2\pi]$ in Cartesian coordinates. How many times does this path wind around the z-axis as s ranges from 0 to 2π? How many times does it wind around the circle $x^2 + y^2 = 4,\ z = 0$?

This recovers the parametrization of the trefoil from the beginning of the term! Here are two views of this curve on the surface of a torus:





Taking other pairs of integers (p, q) the paths $s \mapsto \sigma(ps, qs),\ s \in [0, 2\pi]$ define curves on the torus surface known as (p, q)-torus links (a link is a knot with possibly more than one connected piece).

Some things to ponder: What is the condition on the pair (p, q) so that the (p, q)-torus link is a knot (in other words, has a single connected piece)? What will the path $s \mapsto \sigma(s,\ \sqrt{2}s),\ s \in \mathbb{R}$ look like?