MTHE 227 Problem Set 7

Due Thursday November 03 2016 at the beginning of class

1 (Jacobian of a Linear Map)**.** For $x(u,v) = au + bv$ and $y(u,v) = cu + dv$, show that

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus, the Jacobian of the map $T\colon \mathbb{R}^2_{(u,v)} \to \mathbb{R}^2_{(x,y)}$ given by $(u,v) \mapsto (au + bv,\, cu + dv)$ is everywhere equal to T itself (and, as discussed in lecture, any linear map from $\mathbb{R}^2$ to itself can be written in this form). This fact is consistent with the intuition that the Jacobian of T at (u_0, v_0) is the linear map that best approximates T at (u_0, v_0). (If T is itself linear, then the best linear approximation is itself!)

2 (Geometry of Linear Maps)**.** Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 matrix with $\det A = ad - bc \neq 0$. In linear algebra, one proves that A may be brought to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

by finitely many of the following three operations (called elementary row operations):

(Op. 1) Switching two rows.

(Op. 2) Multiplying every entry of a row by a nonzero number.

(Op. 3) Adding a row to another row.

(More generally, any matrix may be brought to its reduced row-echelon form (rref) by a succession of the above three operations. All matrices with nonzero determinant have the identity matrix as their rref.)

(a) Define the following matrices:

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2(\lambda,1) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2(\lambda,2) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad E_3(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_3(2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Check that multiplying A on the left by:[1]

(i) E_1 switches the two rows of A;

(ii) $E_2(\lambda, 1)$ multiplies every entry of the first row of A by λ;

(iii) $E_2(\lambda, 2)$ multiplies every entry of the second row of A by λ;

[1]Note: Multiplying A on the left by E_i means $E_i \cdot A$.

(iv) $E_3(1)$ adds the second row to the first row; and

(v) $E_3(2)$ adds the first row to the second row.

(b) Conclude from part (a), and the linear algebra fact that A may be brought to the identity matrix by a finite sequence of elementary row operations, that there exists a sequence of matrices $M_1, \ldots, M_r$, with each M_i being one of E_1, $E_2(\lambda, 1)$, $E_2(\lambda, 2)$, $E_3(1)$ or $E_3(2)$, so that

$$M_r\, M_{r-1} \cdots M_2\, M_1\, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) Check that the inverses of the elementary matrices are:

$$E_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2(\lambda,1)^{-1} = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2(\lambda,2)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \quad E_3(1)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_3(2)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Optional Problem. Check that the inverses of elementary matrices may be written in terms of elementary matrices:

$$\begin{aligned} E_1^{-1} &= E_1, \\ E_2(\lambda,1)^{-1} &= E_2\left(\frac{1}{\lambda}, 1\right), \\ E_2(\lambda,2)^{-1} &= E_2\left(\frac{1}{\lambda}, 2\right), \\ E_3(1)^{-1} &= E_2(-1,1)\, E_3(1)\, E_2(-1,1), \\ E_3(2)^{-1} &= E_2(-1,2)\, E_3(2)\, E_2(-1,2). \end{aligned}$$

(d) Recall that a 2×2 matrix defines a linear transformation on $\mathbb{R}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

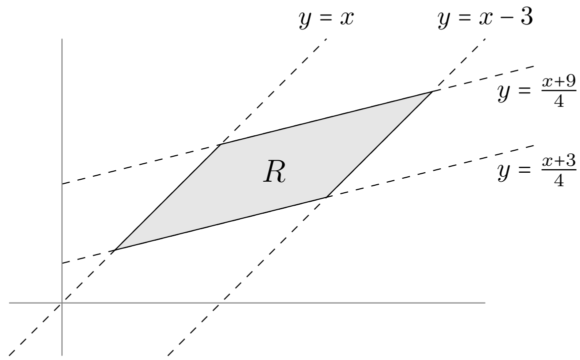
For each of E_1^{-1}, $E_2(\lambda,1)^{-1}$, $E_2(\lambda,2)^{-1}$, $E_3(1)^{-1}$ and $E_3(2)^{-1}$, draw the image of the unit square $[0,1] \times [0,1]$ under the associated linear transformation. Identify each one as being a scaling, shear, or reflection about the diagonal $x = y$.

(e) Conclude that an arbitrary linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ with nonzero determinant may be realized as a composition of finitely many scalings, shears, and reflections about the diagonal.

(f) Draw the image of the unit square under each of the following linear maps, and decompose each of the linear maps into a sequence of scalings, shears, and reflections about the diagonal (row-reduce the matrix, keeping track of the steps!):

$$\text{(i)} \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

3 (Double Integral Over a Parallelogram, Once Again)**.** Recall the parallelogram R with vertices $(1,1)$, $(3,3)$, $(5,2)$, $(7,4)$ from Problem Set 5:



The form of the equations of the boundary lines suggests that

$$u(x,y) = y - x, \qquad v(x,y) = y - \frac{x}{4}$$

is a good change of variables for this problem.

This describes the inverse map $T^{-1}\colon \mathbb{R}^2_{(x,y)} \to \mathbb{R}^2_{(u,v)}$.

(a) What is the region $R^* = T^{-1}(R)$ in $\mathbb{R}^2_{(u,v)}$?

(b) Solve for $x = x(u,v)$ and $y = y(u,v)$ as functions of u and v. (This amounts to finding the inverse of T^{-1}, or, in other words, finding T.)

(c) Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

(d) Compute $\iint_R (y-x)^{2016}\, dA$ by applying the change of variables theorem.