MTHE 227 Problem Set 6

Hand in the solution of the following problem by the beginning of class of *Monday October 31, 2016.*

Note: Problem Set 7 will be due on Thursday November 03, of the same week, so please plan accordingly.

Reminder. In lecture, we have defined the polar coordinate direction vector fields $\mathbf{e}_r$ and $\mathbf{e}_\theta$. These may be expressed in terms of the Cartesian direction vector fields (the latter also known as Cartesian direction vectors, the fields being constant) as [1]

$$\begin{aligned}
\mathbf{e}_r(x,y) &= \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y = \frac{x\mathbf{e}_x + y\mathbf{e}_y}{\sqrt{x^2+y^2}},\\
\mathbf{e}_\theta(x,y) &= -\sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y = \frac{-y\mathbf{e}_x + x\mathbf{e}_y}{\sqrt{x^2+y^2}}.
\end{aligned}$$

Going the other way, we have

$$\begin{aligned}
\mathbf{e}_x(r,\theta) &= \cos\theta\,\mathbf{e}_r(r,\theta) - \sin\theta\,\mathbf{e}_\theta(r,\theta),\\
\mathbf{e}_y(r,\theta) &= \sin\theta\,\mathbf{e}_r(r,\theta) + \cos\theta\,\mathbf{e}_\theta(r,\theta).
\end{aligned}$$

Intuitively, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are steps of unit length in the directions of increasing r and θ, respectively.

1 (Velocity and Acceleration in Polar Coordinates)**.** We have seen that for a path parametrized in polar coordinates by $t \mapsto (r(t),\, \theta(t))$, t in $[a,b]$, the velocity and acceleration vectors are

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(r(t),\theta(t)) + r\frac{d\theta}{dt}\,\mathbf{e}_\theta(r(t),\,\theta(t)), \qquad \text{and}$$

$$\mathbf{a}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(r(t),\theta(t)) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(r(t),\theta(t)).$$

To gain some understanding of the meaning of the various terms in the expression for the acceleration, for each of the following paths: sketch the path, compute the velocity and acceleration in polar coordinates, and sketch the velocity and acceleration vectors at a few points.

(a) (Accelerating Linear Motion) The path $t \mapsto (t^2, \pi/4)$, $t > 0$.

(b) (Uniform Circular Motion) The path $t \mapsto (R,\, 2016\,t)$, $t \in \mathbb{R}$. For this path, check that

$$\|\mathbf{a}(t)\| = \frac{\|\mathbf{v}(t)\|^2}{R} \quad \text{for all } t.$$

(This example is meant to shed some light on the $-r(d\theta/dt)^2$ term.)

[1] The second equalities hold as long as $(x,y) \neq (0,0)$.

(c) (Accelerating Circular Motion) The path $t \mapsto (R,\, 1008t^2),\ t \in \mathbb{R}$.

(d) (Archimedean Spiral) The path $t \mapsto (t,\, t),\ t > 0$.

(One can think of this example as the path followed by a ball rolling radially at unit speed on a platform rotating with unit angular speed, from the reference frame of someone not standing on the platform. It is one of the simplest examples in which the $2\frac{dr}{dt}\frac{d\theta}{dt}$ term is nonzero.)

(e) (A Cardioid) The path $t \mapsto (1 + \cos(t),\ t) = (r(t), \theta(t)),\quad t \in [0, 2\pi]$. This is one possible parametrization of the cardioid from Problem Set 5. Sketch the velocity and acceleration at $t = 0$, $t = \pi/4$, $t = \pi/2$ and $t = \pi$.