MTHE 227 Problem Set 5

Due Thursday October 20 2016 at beginning of class

1 (A Cardioid)**.** Let C be the closed curve in $\mathbb{R}^2$ whose polar coordinates (r, θ) satisfy

$$r = \cos\theta + 1.$$

(a) Sketch the curve C.

(b) Find a parametrization $t \mapsto (r(t), \theta(t)),\ t \in [a, b]$, of C in polar coordinates.

As we have seen in class, for a path parametrized in polar coordinates as in part (b), the arclength of the path can be computed by the expression

$$\int_a^b \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\, dt. \tag{1}$$

(c) Applying (1), or otherwise, show that the arclength of C is equal to 8. (The half-angle formula $|\cos(\frac{\theta}{2})| = \sqrt{\frac{1+\cos(\theta)}{2}}$ may come in useful. You can save yourself from dealing with the absolute value in the half-angle formula by noticing some symmetry in the problem.)

(d) Show that the area enclosed by C is equal to $3\pi/2$.

(e) Convert your parametrization of C found in (b) to Cartesian coordinates (x, y).

(C is a member of a family of curves cut out in polar coordinates by $r = 2a(\cos\theta + 1)$, with a a positive real number, called *cardioids*. The arclength of a general cardioid is $16a$, and its area is $6\pi a^2$. The name derives from the Greek word *kardia*, which translates to "heart".)

2 (Averages of Averages)**.** Two common ways of finding the average of two nonnegative real numbers x and y are the *arithmetic mean*

$$\frac{x+y}{2}$$

and the *geometric mean*

$$\sqrt{xy}.$$

The two means are related by the (very useful) *arithmetic mean-geometric mean inequality*, or *AM-GM inequality* for short:

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Fix a real number $m > 0$. If x and y are chosen independently and uniformly at random from the interval $[0, m]$ (meaning no value is more likely than any other), the expected values of the arithmetic and geometric means are given by

$$\iint_{[0,m]\times[0,m]} \frac{x+y}{2} \cdot \frac{1}{m^2}\, dA \quad \text{and} \quad \iint_{[0,m]\times[0,m]} \sqrt{xy} \cdot \frac{1}{m^2}\, dA,$$

respectively. Compute the two integrals. Are your findings consistent with the AM-GM inequality?

3 (Area of an Ellipse)**.** Find the area of the region R bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, by integrating the constant function 1 over R. What do you find in the case when $a = b$?

You may make use of the following without proof:

$$\int \sqrt{1-t^2}\,dt = \frac{t\sqrt{1-t^2} + \arcsin(t)}{2}.$$

Optional Problem. Integrate $\sqrt{1-t^2}$, by making the trigonometric substitution $t = \sin\theta$.

4. (a) For each of the following regions R, sketch R, and set up the double integral $\iint_R f(x,y)\,dA$ as an iterated integral (or possibly a sum of iterated integrals) in Cartesian coordinates. It is not necessary to evaluate the integral.

(i) The region R to the left of the y-axis and inside the circle $x^2 + y^2 = 1$.

(ii) The region R bounded by the curves $x = y^2 + 3$ and $x = 4y^2$.

(iii) The region R bounded by the parallelogram with vertices $(1,1)$, $(3,3)$, $(5,2)$ and $(7,4)$.

(b) For each of the following, sketch the region of integration, switch the order of integration, and evaluate the integral.

(i) $\displaystyle\int_0^1 \left(\int_0^{\arcsin(x)} y^2\,dy \right) dx$

(ii) $\displaystyle\int_0^1 \left(\int_y^1 e^{x^2}\,dx \right) dy$ (Note: The antiderivative of e^{x^2} cannot be written down in terms of sums, products and powers of the usual functions x, $\cos(x)$, $\sin(x)$, $\exp(x)$, $\log(x)$, ... It has no antiderivative in elementary terms.)

(iii) $\displaystyle\int_{1/2}^1 \left(\int_1^{2y} \frac{\ln x}{x}\,dx \right) dy + \int_1^2 \left(\int_y^2 \frac{\ln x}{x}\,dx \right) dy.$