MTHE 227 Problem Set 1

Due Wednesday September 21 2016 at the beginning of class

1 (Secant and Tangent)**.** This problem describes an interpretation of the trigonometric functions sec and tan in terms of the geometry of the unit circle.

Let C be the unit circle in $\mathbb{R}^2$ centered at the origin. Parametrize an arc of C by

$$\begin{cases} x(\theta) = \cos\theta \\ y(\theta) = \sin\theta \end{cases}, \qquad -\pi/2 < \theta < \pi/2.$$

(a) Draw a picture of this arc. Is the arc traversed clockwise or counterclockwise? Is either endpoint included in the path? Find θ so that $(x(\theta), y(\theta)) = (1, 0)$.

For each $-\pi/2 < \theta < \pi/2$, denote the tangent line to C at $(x(\theta), y(\theta))$ by T_θ, and denote the point of intersection of T_θ with the x-axis by p_θ.

(b) Parametrize the line T_θ.

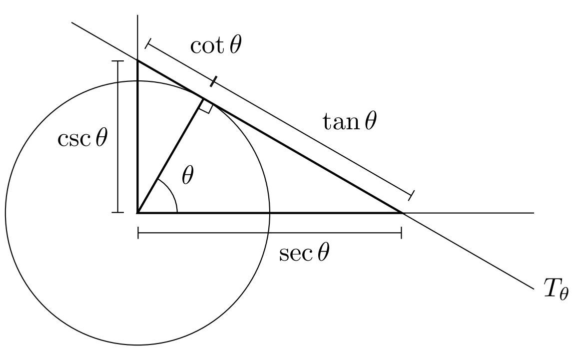
Definitions. For each $-\pi/2 < \theta < \pi/2$, define $\sec\theta$ as the distance of p_θ from the origin, and define $\tan\theta$ as the distance of p_θ from the point of tangency of T_θ with C (please see the picture on the bottom left).

(c) Why can't $\sec(\pi/2)$, $\tan(\pi/2)$, $\sec(-\pi/2)$ and $\tan(-\pi/2)$ be defined similarly?

(d) Show that $p_\theta = (1/\cos\theta, 0)$, and conclude that $\sec\theta = 1/\cos\theta$ and $\tan\theta = \sin\theta/\cos\theta$. Thus, the above definitions of sec and tan are equivalent to the usual analytic ones.[1]

(e) Show that the identity $\tan^2\theta + 1 = \sec^2\theta$ holds, in two ways: first, using the above geometric definitions, and second, using the analytic definitions recovered in part (d).

Optional Problem. Show part (d) without using a parametrization.



The cosecant and cotantent have similar interpretations— keeping the notation above and letting q_θ denote the point of intersection of the tangent line T_θ with the y-axis, $\csc\theta$ and $\cot\theta$ are equal to the distance of q_θ from the origin and point of tangency, respectively (the prefix *co* refers to the complementary angle $\pi/2 - \theta$). Although the sine and cosine turn out to have more importance in mathematics, all trigonometric functions are useful, and their geometric meaning makes identities such as

$$(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta \text{ and}$$

$$\tan(\pi/2 - \theta) = \cot(\theta)$$

more transparent.

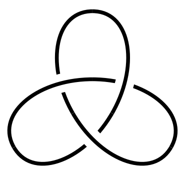
[1] At least in the range $-\pi/2 < \theta < \pi/2$, but the definitions extend quite readily to any $\theta \neq n\pi + \pi/2$.

2 (Intersection of Two Surfaces)**.** The pair of surfaces in $\mathbb{R}^3$ defined by $x^5 + 4y^2 + z = 3x^2y$ and $5x^3 = y + 6$ intersect along a curve C.

(a) Parametrize C. (*Suggestion*: For points on C, express y and z as functions of x.)

(b) Parametrize the tangent line to C at the point $(1, -1, -8)$.

3 (Arclength of the Trefoil Knot)**.** There exist simple closed curves in $\mathbb{R}^3$ that cannot be continuously deformed to the unit circle without introducing self-intersections along the way. Such curves are called *knotted*, and the simplest knotted curve is known as the *trefoil knot*. One possible parametrization of the trefoil knot[2] is given by



$$\begin{cases} x(t) = (2 + \cos 3t)\cos 2t \\ y(t) = (2 + \cos 3t)\sin 2t \\ z(t) = \sin 3t \end{cases}, \quad 0 \le t \le 2\pi.$$

(a) Verify that the arclength of the trace of the given parametrization is given by

$$\int_0^{2\pi} \sqrt{25 + 16\cos(3t) + 4\cos^2(3t)}\, dt.$$

Unfortunately, this integral is too hard (and perhaps even not possible) to find using methods and functions learned about in first year. Nevertheless, it is possible to understand something about the arclength of the trefoil by bounding the integrand above and below:

(b) Show that the integrand satisfies

$$3 \le \sqrt{25 + 16\cos(3t) + 4\cos^2(3t)} \le 3\sqrt{5} \quad \text{for all } t.$$

(c) Apply the general estimate from class to conclude that

$$6\pi \le \text{Arclength of trefoil} \le 6\pi\sqrt{5}.$$

(These three numbers are close to 20, 30 and 40, respectively.)

Optional Problem. We were not very careful in our analysis, and it is possible to find better bounds. Do this, and tell the instructor about them! For example, can you show that

$$8\pi \le \text{Arclength of trefoil} \le (3\sqrt{5} + \sqrt{29})\pi?$$

[2]More precisely, the *positive* trefoil. The trefoil's mirror image is considered to be a different knot, as it cannot be continuously deformed to the positive trefoil (without introducing self-intersections).