Solution to Exercise from Class of Oct. 06

Exercise. *Show that the flux of the vector field*

$$\mathbf{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

is the same across all circles centered at the origin (oriented outward).

Solution. The circle of radius R centered at the origin can be parametrized by

$$t \mapsto (R\cos t,\, R\sin t), \quad t \in [0, 2\pi].$$

First, notice that we can find the outward normal geometrically. Indeed, at any point on the circle, the outward normal points along the position vector of that point. Since the parametrization moves with constant speed R (as we have seen in class), $\mathbf{n}_+(t) = (R\cos t,\, R\sin t)$ (since the parametrization goes counterclockwise, the outward normal is $\mathbf{n}_+$).

Of course, we can also compute the normal:

$$\begin{aligned}\mathbf{v}(t) &= (-R\sin t,\, R\cos t)\\ \mathbf{n}_+(t) &= (y'(t), -x'(t)) = (R\cos t,\, R\sin t),\end{aligned}$$

as expected.

Now,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}_+(t) = \left(\frac{R\cos t}{R^2\cos^2 t + R^2\sin^2 t}, \frac{R\sin t}{R^2\cos^2 t + R^2\sin^2 t}\right) \cdot (R\cos t,\, R\sin t) = 1,$$

so that the flux across the circle is equal to

$$\int_{\text{Circle of Radius } R} \mathbf{F}\cdot\hat{\mathbf{n}}\, ds = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{n}_+(t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi,$$

independent of R!

The magnitude of $\mathbf{F}$ decreases radially at a rate that just compensates for the increase in the circumference of the circle.