MTHE 227 Final Exam

December 10, 2016
Queen's University Applied Science
Department of Mathematics and Statistics
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Instructions: This exam has nine problems; each probem is worth ten points. For problems with multiple parts, the worth of the individual parts is stated before the statement of the problem.

The exam is three hours in length.

Write your answers in the booklets provided. Hand in both the booklets and the question paper.

To receive full credit, you must justify your answers. Answers with little or no justification will receive little or no credit.

Calculators, data sheets, notes, and other aids are not permitted. A short reference sheet is provided on the last page.

Please Note: Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

Student Number: ______________________

1	2	3	4	5	6	7	8	9	Total

Problem 1 (4+4+2 points). (a) Find constants a and b that make the following vector field conservative:

$$\mathbf{F}(x,y,z) = \left(\, 2xy + z^2,\ ax^2 + z,\ y + bxz \,\right).$$

(b) For these values of a and b, find a function $f : \mathbb{R}^3 \to \mathbb{R}$ so that $\mathbf{F} = \mathrm{grad}(f)$.

(c) For the same values of a and b, find the equation $g(x,y,z) = c$ of a surface S in $\mathbb{R}^3$ with the property that for any pair of points P and Q on S, and any path C from P to Q,

$$\int_C \mathbf{F} \cdot \mathbf{dr} = 0$$

(such a surface is not unique).

Problem 2 (5+5 points). (a) Let $\mathbf{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$ be a vector field on $\mathbb{R}^3$. If each of the component functions F_i have continuous second partial derivatives, verify the identity

$$\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0.$$

(b) Let f be a differentiable function on $\mathbb{R}^3$, and let $f\mathbf{F}$ denote the vector field

$$f\mathbf{F}(x,y,z) := (\, f(x,y,z)\,F_1(x,y,z),\ f(x,y,z)\,F_2(x,y,z),\ f(x,y,z)\,F_3(x,y,z)\,).$$

Verify the identity

$$\mathrm{div}(f\mathbf{F}) = \mathrm{grad}(f) \cdot \mathbf{F} + f\ \mathrm{div}(\mathbf{F}).$$

Problem 3 (4+6 points). The cylindrical-to-Cartesian coordinate transformations are:

$$\begin{aligned} x(r,\theta,z) &= r\ \cos(\theta), \\ y(r,\theta,z) &= r\ \sin(\theta), \\ z(r,\theta,z) &= z. \end{aligned}$$

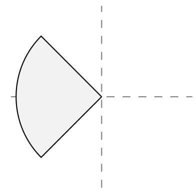
(a) Verify that $\left| \det \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} \right| = r$, where $|\cdot|$ denotes the absolute value.

(b) Let R be the region in $\mathbb{R}^3_{(x,y,z)}$ described by the inequalities $z \geq x^2 + y^2,\ x \geq 0,\ y \geq 0$, and $z \leq 4$. Compute

$$\iiint_R \frac{3xy^2 z}{\sqrt{x^2 + y^2}}\, dV.$$

Problem 4 (10 points)**.** Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y) = (-x^2y,\, xy^2)$, and $C = C_1 + C_2 + C_3$ a closed piecewise curve with the following pieces:

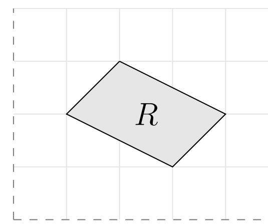
$$\begin{cases} C_1 : \text{Line segment from } (0,0) \text{ to } (-2,2), \\ C_2 : \text{Circle arc from } (-2,2) \text{ to } (-2,-2), \\ C_3 : \text{Line segment from } (-2,-2) \text{ to } (0,0). \end{cases}$$



Compute the work $\int_C \mathbf{F} \cdot \mathbf{dr}$ done by $\mathbf{F}$ around C.

Problem 5 (10 points)**.** Let R be the parallelogram with vertices $(1,2)$, $(2,3)$, $(4,2)$ and $(3,1)$ in $\mathbb{R}^2$. Using the change of variables

$$\begin{cases} u = 2y + x \\ v = y - x \end{cases},$$



compute the integral

$$\iint_R (y - x) e^{2y+x} \, dA.$$

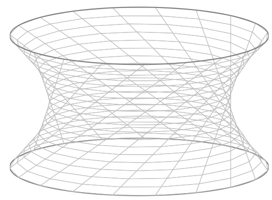
Problem 6 (10 points)**.** Let C be the curve in the xz-plane described by $(x-3)^2 + (z-5)^2 = 4$, $x \geq 3$. Find the surface area of the surface of revolution of C about the z-axis. (You may, but do not have to, use Pappus' theorem.)

Problem 7 (10 points)**.** Let R be the region in $\mathbb{R}^3$ bounded by the two spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 16$ and contained in the half-space $y \geq 0$. Let $S = \partial R$ be the boundary of R, oriented outward. Find the flux $\iint_S \mathbf{F} \cdot \mathbf{dS}$ of the vector field $\mathbf{F}(x, y, z) = (xy^2,\, yz^2,\, zx^2)$ through S.

Problem 8 (2+2+6 points)**.** Let S be the surface parametrized by

$$(u, v) \mapsto (\cos(u) - v\sin(u),\ \sin(u) + v\cos(u),\ v), \quad u \in [0, 2\pi],\ v \in [-1, 1].$$

This surface is called (a part of) a hyperboloid of one sheet. The boundary of S consists of two circles.



(a) Check that every point of S satisfies the equality

$$x^2 + y^2 - z^2 = 1.$$

Let $\mathbf{F}(x, y, z) = (xz - 1,\ y - yz,\ -z)$ and $\mathbf{G}(x, y, z) = (yz,\ z,\ xyz)$.

(b) Check that $\operatorname{curl} \mathbf{G} = \mathbf{F}$.

(c) Find the flux $\iint_S \mathbf{F} \cdot \mathbf{dS}$ of $\mathbf{F}$ through S, with normals oriented away from the z-axis.

Problem 9 (5+5 points)**.** Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right),$$

with domain

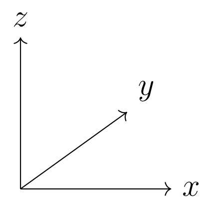
$$D = \left\{ (x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0) \right\}.$$

(a) Compute the flux $\int_{C_\epsilon} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$ of $\mathbf{F}$ across the circle C_ϵ of radius ϵ centered at the origin (and with normal pointing away from the origin). Check that the result is independent of the radius ϵ.

(b) Let Γ be a simple closed curve in $\mathbb{R}^2$, and suppose that the origin is contained in the region bounded by Γ. Argue, applying the extended Green's theorem for flux, that the flux $\int_\Gamma \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$ of $\mathbf{F}$ across Γ (and with the normals pointing away from the origin) is equal to the number found in part (a).

Reference Sheet

Right-Handed Coordinate System.



Trigonometric identities.

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta). \qquad \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}.$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta). \qquad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}.$$

Jacobians.

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}, \qquad \frac{\partial(x,y,z)}{\partial(u,v,w)} := \begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{pmatrix}.$$

Cylindrical Coordinates (r, θ, z)**.**

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z. \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r.$$

Spherical Coordinates (ρ, ϕ, θ)**.**

$$x = \rho\cos(\theta)\sin(\phi), \quad y = \rho\sin(\theta)\sin(\phi), \quad z = \rho\cos(\phi). \qquad \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin(\phi).$$

Equivalent Notations for Div, Grad and Curl.

$$\mathrm{div}(\mathbf{F}) = \nabla\cdot\mathbf{F}, \qquad \mathrm{grad}(f) = \nabla f, \qquad \mathrm{curl}(\mathbf{F}) = \nabla\times\mathbf{F}.$$